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Constrained supermanifolds for AdS M-theory backgrounds*

Pietro Fré[ad] and Pietro Antonio Grassi[bcd]

[a] *Dipartimento di Fisica Teorica, Università di Torino, via P. Giuria 1, I-10125 Torino, Italy*
[b] *Centro Studi e Ricerche E. Fermi, Compendio Viminale, I-00184, Roma, Italy,*
[c] *DISTA, Università del Piemonte Orientale, via Bellini 25/G, Alessandria, 15100, Italy*
[d] *INFN, sezione di Torino, via P. Giuria 1, I-10125 Torino, Italy*
E-mail: fre@to.infn.it, pgrassi@cern.ch

Abstract: A long standing problem is the supergauge completion of $\mathrm{AdS}_4 \times (\mathcal{G}/\mathcal{H})_7$ or $\mathrm{AdS}_5 \times (\mathcal{G}/\mathcal{H})_5$ backgrounds which preserve less then maximal supersymmetry. In parallel with the supersolvable realization of the $\mathrm{AdS}_4 \times \mathbb{S}^7$ background based on κ-symmetry, we develop a technique which amounts to solving the above-mentioned problem in a way useful for pure spinor quantization for supermembranes and superstrings. Instead of gauge fixing some of the superspace coordinates using κ-symmetry, we impose an additional constraint on them reproducing the simplifications of the supersolvable representations. The constraints are quadratic, homogeneous, $\mathrm{Sp}(4,\mathbb{R})$-covariant, and consistent from the quantum point of view in the pure spinor approach. Here we provide the geometrical solution which, in a subsequent work, will be applied to the membrane and the superstring sigma models.

Keywords: Supergravity Models, Superspaces, Superstrings and Heterotic Strings, BRST Symmetry.

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Contents

1. Introduction

One of the most interesting progress in the theory of supermembranes is the quantization by using the pure spinor technique [1]. It provides a quantum model (interacting) where the kappa symmetry is gauge fixed and a BRST is provided. Using the BRST operators one can compute the cohomology and the spectrum. Unfortunately, the interacting world-volume action does not allow a simple analysis of the complete spectrum and only the massless sector can be studied by using the target space symmetries. Nevertheless, the main advantage is a complete superspace description of the theory in terms of vielbeins, gravitinos and the superfield generalization of the 3-form of 11-dimensional supergravity [2, 3]. Recently in [4], we have shown that there is a deep relation between the pure spinor BRST symmetry [5, 6] and the Free Differential Algebra of 11 supergravity and we have used these facts to obtain a complete algebraic derivation of the BRST symmetry and of the symmetries of the model. The resulting action has a manifest supersymmetry and it depends on the supergravity background superfields. Those superfields are obtained from the FDA by gauge completing the superfield starting from a given bosonic background which satisfies the equations of motion. (We have to remind the reader that the FDA's for 11-dimensional supergravity discussed in section 2 imply the equations of motion).

However, to solve the FDA for a given background is not a trivial task and the complete superfield is needed in order to compute amplitudes in presence of a given background. In practice one needs a superfield only up to a certain power in the fermionic coordinates. The reason is that the coefficients of higher powers are simply ordinary derivatives of the lowest components and they do no carry new information. Nevertheless, those coefficients enter the computation of amplitudes and we need a method to reconstruct a complete superfield in terms of the bosonic solution. There are on the market several techniques, see for example [7 − 10] just to quote some of them adapted to our problem. These techniques start from a very general setting and they provide an iterative reconstruction method, which unfortunately hides completely the geometry behind the solution. We take a different perspective: we start from a solution with some supersymmetries (in our case, from the 4-dimensional point of view we take the supersymmetric models with $\mathcal{N} = 8, 3, 1$) and some relevant isometries and we try to build a complete superfield solution respecting these symmetries. The rheonomic parametrizations of FDA.s are integrable by construction and the consistency conditions are just the equations of motion [3]. Therefore we need to start from an on-shell background solution and we are guaranteed that the solution exists. The best way to find complete solutions of the FDA is terms of a super-Lie algebra and of its Maurer-Cartan forms. As will be discussed in next sections, one starts from the

Killing spinor of the bosonic solution and he reconstructs the gravitino fields by "pairing" the Killing spinors of the bosonic submanifold with fermionic Maurer-Cartan forms of the underlying algebra. Then, by inserting the gravitino field in the FDA and using the relations between the Maurer-Cartan forms dictated by the Lie superalgebra, one finds that the gravitinos satisfy their own equations. In the same way one can modfify the bosonic supervielbein by adding the bosonic Maurer-Cartan forms and, by inserting it into the FDA equations, one finds all correct pieces. This technique permits a direct complete solution of the gauge completion only for supergroups or supercosets. It does not work that simply in the case of less supersymmetry of the background, and some modifications are needed.

First, one needs to study the obstruction that prevents one from getting a complete solution as a supergroup or a supercoset. This is parameterized by the Weyl tensor which is obtained by commuting two covariant derivatives. Second, one finds that some of the structures of the supercoset technique can still be used. For example, one can organize the fermionic coordinates in two sectors: 1) those related to the linear realization of supersymmetry (the unbroken supersymmetries) and 2) the remaining set related to the broken supersymmetries, and the most convenient method seems to follow very closely the supercoset solution. We assume that the fermionic coordinates are organized according to a pure fermionic supercoset and we construct the gravitinos by pairing the Killing spinors and some other spinor (needed to span a complete basis of sections of the spinor bundle over the bosonic submanifold) with the Maurer-Cartan forms. The violation of the FDA can be compensated by adding to the gravitions and to other superfield additional pieces. These pieces can be taken automatically into account, by promoting the Maurer-Cartan forms to *gauged Maurer-Cartan forms*. This yields an additional term in the vielbein equation which can be reabsorbed into a redefintion of the spin connection. In this way the procedure can be iterated (even if it will not be pursued here further) and one lands with a complete superfield construction.

Fortunately, there is an interesting alternative to the iterative solution. This procedure has been developed in [11, 12] and used in several applications (see for example [13]) and it is based on the supersolvable realization of the supercoset $\mathrm{Osp}(8|4)/\mathrm{SO}(1,3) \times \mathrm{Sp}(4,\mathbb{R})$ in the case of D=11 supergravity and of $\mathrm{SU}(2,2|4)/\mathrm{SO}(1,4)\times\mathrm{SO}(5)$ for the superstring. Using the κ-symmetry one can gauge some coordinates of the superspace to zero and write the Maurer-Cartan equations only in terms of the reduced superspace. This has the advantage to fix the gauge symmetry and to simplify the Maurer-Cartan forms drastically. Specifically it turns out that after this gauge fixing, they are just quadratic in the θ-coordinates. In this way, the problem of resumming the complete dependence of the fermionic coordinates is avoided and the gauged Maurer-Cartan equations already suffice to solve the problem of the gauge completion. Indeed, only a remaining additional piece of contorsion must be added in order to compensate the non-vanishing of the Weyl tensor.

This for what concerns the models with κ-symmetry where the gauge completion can be provided.[1] However, we notice that the same simplification can be achieved by imposing

[1]There might be some difficulties to reach the κ-symmetry gauge using the supersolvable realizations as observed in [52, 53]. However, we can observe that the usage of pure spinor technology instead of κ-

a constraint on the fermionic coordinates. In the case of $\mathrm{Osp}(8|4)/\mathrm{SO}(1,3)\times\mathrm{Sp}(4,\mathbb{R})$ is

$$\Theta^x_A \epsilon_{xy} \Theta^y_B = 0\,. \tag{1.1}$$

Here the indices A, B run over $1,\ldots,8$ and the indices x, y over $1,\ldots,4$. The equation is symmetric in the SO(8) indices, it is homogeneous of degree two in the scaling of Θ's, is quadratic and it is $\mathrm{Sp}(4,\mathbb{R})$ covariant which means that it does not spoil the isometries of the AdS_4 manifold. It will be shown in the text that these constraints yield the same simplification of the supersolvable realization of the supercoset, and in particular the κ-symmetry gauge adopted in [11, 12] is a solution of these new constraints. However, in the case of Green-Schwarz type of models these constraints are not consistent with the canonical quantization of the model. This is due to fact that in the canonical quantization the Θ's satisfy a Clifford algebra and the above constraints are not consistent. On the other side, using the pure spinor formalism the commutation relations among Θ's vanish (they have a non-vanishing commutation relations with the conjugate momenta, see for example [4]) and the constraints are consistent. In addition, they have the same dignity of the pure spinor constraints and they can be treated on the same footing. (We also mention that quadratic constraints for the supercoordinates appeared also in [14−16] and in [17]. In [18], which is based on pure spinor formulation of BRST symmetry [19, 20], quadratic constraints for anticommuting ghosts have been discussed.)

In this way, we can use the advantages of the supersolvable description of the background in order to derive pure spinor sigma models for supermembrane and superstrings. This can be useful for maximal supersymmetric background and for less than maximal supersymmetric spaces.

The paper is organized as follows. In section 2 and section 3, we give some details about compactifications of the bosonic background of the form $\mathrm{AdS}_4\times\mathcal{G}/\mathcal{H}$, free differential algebras and some notations. In section 4, we recall the geometry of the spinor bundle and the holonomy tensor. In section 5 we discuss some property of the supergroup $\mathrm{Osp}(\mathcal{N}|4)$ and its Maurer-Cartan forms. Finally, we discuss the gauging and we discuss the solution to the first order. Then, we consider two examples in section 9. Some additional material is contained in the appendices.

2. The super FDA of D=11 supergravity

Let us begin by writing the complete set of curvatures defining the complete FDA of $D = 11$ supergravity. As usual this FDA is the semidirect sum of a minimal algebra with a contractible algebra:

$$\mathbb{A} = \mathbb{M} \uplus \mathbb{C} \tag{2.1}$$

the curvatures being the contractible generators $\mathbb{C}$. By setting them to zero we retrieve, according to Sullivan's first theorem, the minimal algebra $\mathbb{M}$. This latter, according to

symmetry might overcome these problems and supersolvable realizations can be used not only for string theory on anti-de Sitter background, but also for membrane. Our motivations are indeed focused on pure spinor models instead of κ-invariant solutions. The formulation using constrained supermanifold represents an alternative to gauge fixing κ-symmetry.

Sullivan's second theorem, is explained in terms of cohomology of the super Lie subalgebra $\mathbb{G} \subset \mathbb{M}$, spanned by the 1-forms. In this case $\mathbb{G}$ is just the $D = 11$ superPoincaré algebra spanned by the following 1-forms:

1. the vielbein $V^{\underline{a}}$
2. the spin connection $\omega^{\underline{ab}}$
3. the gravitino Ψ

where the underlined indices $\underline{a}, \underline{b}, \ldots$ run on eleven values and are vector indices of $\mathrm{SO}(1,10)$. The gravitino Ψ is a fermionic one-form (hence commuting) assigned to the 32-component Majorana spinor representation of $\mathrm{SO}(1,10)$:

$$C\overline{\Psi}^T = \Psi \quad ; \quad \overline{\Psi} \equiv \Psi^\dagger \Gamma_0 \tag{2.2}$$

The higher degree generators of the minimal FDA $\mathbb{M}$ are:

1. the bosonic 3-form $\mathbf{A}^{[\mathbf{3}]}$
2. the bosonic 6-form $\mathbf{A}^{[\mathbf{6}]}$.

The complete set of curvatures is given below ([21, 22]):

$$\begin{aligned}
T^{\underline{a}} &= \mathcal{D}V^{\underline{a}} - \mathrm{i}\frac{1}{2}\,\overline{\Psi} \wedge \Gamma^{\underline{a}}\,\Psi \\
R^{\underline{ab}} &= d\omega^{\underline{ab}} - \omega^{\underline{ac}} \wedge \omega^{\underline{cb}} \\
\rho &= \mathcal{D}\Psi \equiv d\Psi - \frac{1}{4}\,\omega^{\underline{ab}} \wedge \Gamma_{\underline{ab}}\,\Psi \\
\mathbf{F}^{[\mathbf{4}]} &= d\mathbf{A}^{[\mathbf{3}]} - \frac{1}{2}\,\overline{\Psi} \wedge \Gamma_{\underline{ab}}\,\Psi \wedge V^{\underline{a}} \wedge V^{\underline{b}} \\
\mathbf{F}^{[\mathbf{7}]} &= d\mathbf{A}^{[\mathbf{6}]} - 15\,\mathbf{F}^{[\mathbf{4}]} \wedge \mathbf{A}^{[\mathbf{3}]} - \frac{15}{2}\,V^{\underline{a}} \wedge V^{\underline{b}} \wedge \bar{\Psi} \wedge \Gamma_{\underline{ab}}\,\Psi \wedge \mathbf{A}^{[\mathbf{3}]} \\
&\quad -\mathrm{i}\,\frac{1}{2}\,\overline{\Psi} \wedge \Gamma_{\underline{a_1 \ldots a_5}}\,\Psi \wedge V^{\underline{a_1}} \wedge \ldots \wedge V^{\underline{a_5}}
\end{aligned} \tag{2.3}$$

From their very definition, by taking a further exterior derivative one obtains the Bianchi identities:

$$\begin{aligned}
\mathcal{D}R^{\underline{ab}} &= 0 \\
\mathcal{D}T^{\underline{a}} + R^{\underline{a}}{}_{\underline{b}} \wedge V^{\underline{b}} + \bar{\Psi} \wedge \Gamma^{\underline{a}}\rho &= 0 \\
\mathcal{D}\rho + \frac{1}{4}R^{\underline{ab}} \wedge \Gamma_{\underline{ab}}\Psi &= 0\,, \\
d\mathbf{F}^{[\mathbf{4}]} - \bar{\Psi}\Gamma_{\underline{ab}} \wedge \rho \wedge V^{\underline{a}} \wedge V^{\underline{b}} - \bar{\Psi} \wedge \Gamma_{\underline{ab}}\Psi \wedge V^{\underline{a}} \wedge T^{\underline{b}} &= 0
\end{aligned} \tag{2.4}$$

The dynamical theory is defined, according to the general constructive scheme of supersymmetric theories, by the principle of rheonomy (see [23]) implemented into Bianchi identities. Indeed there is a unique rheonomic parametrization of the curvatures which solves the Bianchi identities and it is the following one:

$$
\begin{aligned}
T^{\underline{a}} &= 0 &(2.5)\\
\mathbf{F}^{[4]} &= F_{\underline{a_1 \dots a_4}}\, V^{\underline{a_1}} \wedge \dots \wedge V^{\underline{a_4}} &(2.6)\\
\mathbf{F}^{[7]} &= \frac{1}{84} F^{\underline{a_1 \dots a_4}}\, V^{\underline{b_1}} \wedge \dots \wedge V^{\underline{b_7}}\, \epsilon_{\underline{a_1 \dots a_4 b_1 \dots b_7}} &(2.7)\\
\rho &= \rho_{\underline{a_1 a_2}}\, V^{\underline{a_1}} \wedge V^{\underline{a_2}} + \mathrm{i}\frac{1}{3}\left(\Gamma^{\underline{a_1 a_2 a_3}}\Psi \wedge V^{\underline{a_4}} - \frac{1}{8}\Gamma^{\underline{a_1 \dots a_4 m}}\,\Psi \wedge V^{\underline{m}}\right) F^{\underline{a_1 \dots a_4}} &(2.8)\\
R^{\underline{ab}} &= R^{\underline{ab}}{}_{\underline{cd}}\, V^{\underline{c}} \wedge V^{\underline{d}} + \mathrm{i}\,\overline{\rho}_{\underline{mn}}\left(\frac{1}{2}\Gamma^{\underline{abmnc}} - \frac{2}{9}\Gamma^{\underline{mn[a}}\,\delta^{\underline{b]c}} + 2\,\Gamma^{\underline{ab[m}}\,\delta^{\underline{n]c}}\right)\Psi \wedge V^{\underline{c}}\\
&\quad + \overline{\Psi} \wedge \Gamma^{\underline{mn}}\,\Psi\, F^{\underline{mnab}} + \frac{1}{24}\overline{\Psi} \wedge \Gamma^{\underline{abc_1 \dots c_4}}\,\Psi\, F^{\underline{c_1 \dots c_4}} &(2.9)
\end{aligned}
$$

The expressions (2.5)–(2.9) satisfy the Bianchi equations provided the space-time components of the curvatures satisfy the following constraints

$$
\begin{aligned}
0 &= \mathcal{D}_{\underline{m}} F^{\underline{m c_1 c_2 c_3}} + \frac{1}{96}\,\epsilon^{\underline{c_1 c_2 c_3 a_1 a_8}}\, F_{\underline{a_1 \dots a_4}}\, F_{\underline{a_5 \dots a_8}}\\
0 &= \Gamma^{\underline{abc}}\,\rho_{\underline{bc}}\\
R^{\underline{am}}{}_{\underline{cm}} &= 6\, F^{\underline{a c_1 c_2 c_3}}\, F^{\underline{b c_1 c_2 c_3}} - \frac{1}{2}\,\delta^{\underline{a}}_{\underline{b}}\, F^{\underline{c_1 \dots c_4}}\, F^{\underline{c_1 \dots c_4}} \qquad (2.10)
\end{aligned}
$$

which are the space-time field equations.

2.1 Other relevant implications of the Bianchi identities

For later use it is convenient to rewrite eqs. (2.9) in a slightly more compact form, namely:

$$
\begin{aligned}
T^{\underline{a}} &\equiv 0\,,\\
R^{\underline{ab}} &\equiv R^{\underline{ab}}{}_{\underline{mn}}\, V^{\underline{m}} \wedge V^{\underline{n}} + \bar{\Theta}^{\underline{c}\,|\,\underline{ab}}\,\Psi \wedge V_{\underline{c}} + \overline{\Psi} \wedge S^{\underline{ab}}\,\Psi\,,\\
\rho &\equiv \rho_{\underline{ab}}\, V^{\underline{a}} \wedge V^{\underline{b}} + F_{\underline{a}}\,\Psi \wedge V^{\underline{a}}\,,\\
\mathbf{F}^{[4]} &\equiv F_{\underline{b_1} \dots \underline{b_4}} V^{\underline{b_1}} \wedge \dots \wedge V^{\underline{b_4}}\,. &(2.11)
\end{aligned}
$$

where we have defined the following spinor and the following matrices:

$$
\begin{aligned}
\bar{\Theta}^{\underline{c}\,|\,\underline{ab}} &= \mathrm{i}\,\overline{\rho}_{\underline{mn}}\left(\frac{1}{2}\Gamma^{\underline{abmnc}} - \frac{2}{9}\Gamma^{\underline{mn[a}}\,\delta^{\underline{b]c}} + 2\,\Gamma^{\underline{ab[m}}\,\delta^{\underline{n]c}}\right)\\
&= -\mathrm{i}\,\overline{\rho}_{\underline{ab}}\,\Gamma_{\underline{c}} + 2\,\mathrm{i}\,\overline{\rho}_{\underline{c[a}}\,\Gamma_{\underline{b]}} &(2.12)\\
F_{\underline{a}} &= T_{\underline{a}}{}^{\underline{b_1 b_2 b_3 b_4}} F_{\underline{b_1 b_2 b_3 b_4}}\,, &(2.13)\\
S^{\underline{ab}} &= F^{\underline{abcd}}\Gamma_{\underline{cd}} + \frac{1}{24} F_{\underline{c_1 \dots c_4}}\Gamma^{\underline{abc_1 \dots c_4}}\,, &(2.14)
\end{aligned}
$$

and where where we have used the following abbreviation as in [24]:

$$
T_{\underline{a}}{}^{\underline{b_1 b_2 b_3 b_4}} = -\frac{\mathrm{i}}{24}\left(\Gamma^{\underline{b_1 b_2 b_3 b_4}}{}_{\underline{a}} + 8\,\delta_{\underline{a}}{}^{\underline{[b_1}}\Gamma^{\underline{b_2 b_3 b_4]}}\right)\,. \qquad (2.15)
$$

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In eq. (2.12) the equality of the first with the second line follows from the gravitino field equation, namely the second of eqs. (2.10). This latter implies that the spinor tensor $\rho_{\underline{ab}}$ is an irreducible representation $\left(\frac{3}{2}\,,\,\frac{3}{2}\,,\,\frac{1}{2}\,,\,\frac{1}{2}\,,\,\frac{1}{2}\right)$ of SO(1, 10), *i.e*:

$$\Gamma^{\underline{m}}\,\rho_{\underline{am}} = 0 \tag{2.16}$$

As we demonstrate later on the most important relations to be extracted from Bianchi identities, besides the rheonomic parametrization, concerns the spinor derivatives of the curvature superfield. This latter is determined from the expansion of the inner components of the 4-form field strength $F_{\underline{a_1 \dots a_4}}$. From the last of eqs. (2.4) we obtain:

$$\mathcal{D}_{\underline{\alpha}} F_{\underline{abcd}} = (\Gamma_{[\underline{ab}}\rho_{\underline{cd}]})_{\underline{\alpha}}\,, \tag{2.17}$$

where the spinor derivative is normalized according to the definition:

$$\mathcal{D}\,F_{\underline{abcd}} \equiv \overline{\Psi}^{\underline{\alpha}}\,\mathcal{D}_{\underline{\alpha}} F_{\underline{abcd}} + V^{\underline{m}}\mathcal{D}_{\underline{m}}\,F_{\underline{abcd}} \tag{2.18}$$

This shows that the gravitino field strength appears at first order in the θ-expansion of the curvature superfield. Next we consider the spinor derivative of the gravitino field strength itself. Using the normalization which streams from the following definition:

$$\mathcal{D}\,\rho_{\underline{ab}} = \mathcal{D}_{\underline{c}}\,\rho_{\underline{ab}}\,V^{\underline{c}} + K_{\underline{ab}}\,\Psi \tag{2.19}$$

we obtain:

$$K_{\underline{ab}} = -\frac{1}{4}\,R^{\underline{mn}}{}_{\underline{ab}}\,\Gamma_{\underline{mn}} + \mathcal{D}_{[\underline{a}}\,F_{\underline{b}]} + \frac{1}{2}\,\left[F_{\underline{a}}\,,\,F_{\underline{b}}\right] \tag{2.20}$$

The tensor-matrix $K_{\underline{ab}}$ is of key importance in the discussion of compactifications. If it vanishes on a given background it means that the gravitino field strength can be consistently put to zero to all orders in θ.s and on its turn this implies that the 4-field strength can be chosen constant to all orders in θ.s This is the case of maximal unbroken supersymmetry. In this case all curvature components of the Free Differential Algebra can be chosen constant and we have a superspace whose geometry is purely described by Maurer Cartan forms of some super coset.

On the other hand if $K_{\underline{ab}}$ does not vanish this implies that both $\rho_{\underline{ab}}$ and $F_{\underline{abcd}}$ have some non trivial θ-dependence and cannot be chosen constant. In this case the geometry of superspace is not described by simple Maurer Cartan forms of some supercoset, since the curvatures of the FDA are not pure constants. This is the case of fully or partially broken SUSY and it is the case we want to explore. In the the $\mathrm{AdS}_4 \times (\mathrm{G/H})_7$ compactifications it will turn out that the matrix K_{ab} is related to the holonomy tensor of the internal manifold $(\mathrm{G/H})_7$.

Let us finally work out the spinor derivative of the Riemann tensor. Defining:

$$\mathcal{D}\,R^{\underline{ab}}{}_{\underline{mn}} = \mathcal{D}_{\underline{p}}\,R^{\underline{ab}}{}_{\underline{mn}}\,V^{\underline{p}} + \overline{\Psi}\,\Lambda^{\underline{ab}}{}_{\underline{mn}} \tag{2.21}$$

from the first of eqs. (2.4) we obtain:

$$\Lambda^{\underline{ab}}{}_{\underline{mn}} = \left(\mathcal{D}_{[\underline{m}} - \overline{F}_{[\underline{m}}\right)\Theta_{\underline{n}]}{}^{|\underline{ab}} + 2\,S^{\underline{ab}}\,\rho_{\underline{mn}} \tag{2.22}$$

where we have introduced the notation:

$$\begin{aligned}\Theta^{\underline{n}\,|\,\underline{ab}} &= C\,\left(\bar{\Theta}^{\underline{n}\,|\,\underline{ab}}\right)^T \;=\; \mathrm{i}\,\Gamma_{\underline{c}}\,\rho_{\underline{ab}} \;-\; 2\,\mathrm{i}\,\Gamma_{[\underline{a}}\,\rho_{\underline{b}]\underline{c}} \\ \overline{F}_{\underline{a}} &= C\,(F_a)^T\,C^{-1} \;=\; \frac{\mathrm{i}}{24}\left(\Gamma^{\underline{b_1 b_2 b_3 b_4}}{}_{\underline{a}} \;-\; 8\,\delta_{\underline{a}}{}^{[\underline{b_1}}\Gamma^{\underline{b_2 b_3 b_4}]}\right)\,F_{\underline{b_1 b_2 b_3 b_4}}\end{aligned} \tag{2.23}$$

The matrix $K_{\underline{ab}}$ and the spinor $\Lambda^{\underline{ab}}{}_{\underline{mn}}$ are the crucial objects we are supposed to compute in each compactification background.

3. Compactifications of M-theory on AdS$_4$ $\times$ $\mathcal{M}_7$ backgrounds

We are interested in compactified backgrounds where the 11-dimensional bosonic manifold is of the form:

$$\mathcal{M}_{11} = \mathcal{M}_4 \,\times\, \mathcal{M}_7 \tag{3.1}$$

$\mathcal{M}_4$ denoting a four-dimensional maximally symmetric manifold whose coordinates we denote x^μ and $\mathcal{M}_7$ a 7-dimensional compact manifold whose parameters we denote y^I. Furthermore we assume that in any configuration of the compactified theory the eleven dimensional vielbein is split as follows:

$$V^{\underline{a}} = \begin{cases} V^r \;=\; E^r(x) & ; \quad r = 0,1,2,3 \\ V^\alpha \;=\; \Phi^\alpha{}_\beta(x)\,\left(e^\beta + W^\beta(x)\right) & ;\; \alpha,\beta = 4,5,6,7,8,9,10 \end{cases} \tag{3.2}$$

where $E^r(x)$ is a purely x-dependent 4-dimensional vielbein, $W^\alpha(x)$ is an x-dependent 1-form on x-space describing the Kaluza Klein vectors and the purely x–ependent 7×7 matrix $\Phi^\alpha{}_\beta(x)$ encodes part of the scalar fields of the compactified theory, namely the internal metric moduli. From these assumptions it follows that the bosonic field strength is expanded as follows:

$$\begin{aligned}\mathbf{F}^{[4]}_{(Bosonic)} &\equiv F^{[4]}(x) \,+\, F^{[3]}_\alpha(x)\,\wedge\,V^\alpha + F^{[2]}_{\alpha\beta}(x)\,\wedge\,V^\alpha\,\wedge\,V^\beta \\ &\quad + F^{[1]}_{\alpha\beta\gamma}(x)\,\wedge\,V^\alpha\,\wedge\,V^\beta\,\wedge\,V^\gamma \,+\, F^{[0]}_{\alpha\beta\gamma\delta}(x)\,\wedge\,V^\alpha\,\wedge\,V^\beta\,\wedge\,V^\gamma\,\wedge\,V^\delta\end{aligned} \tag{3.3}$$

where $F^{[p]}_{\alpha_1\ldots\alpha_{4-p}}(x)$ are x-space p-forms depending only on x.

In bosonic backgrounds with a space-time geometry of the form (3.1), the family of configurations (3.2) must satisfy the condition that by choosing:

$$E^r \;=\; \text{vielbein of a maximally symmetric 4-dimensional space time} \tag{3.4}$$

$$\Phi^I{}_J(x) \;=\; \delta^I{}_J \tag{3.5}$$

$$W^I \;=\; 0 \tag{3.6}$$

$$F^{[3]}_I(x) \;=\; F^{[2]}_{IJ}(x) \;=\; F^{[1]}_{IJK}(x) \;=\; 0 \tag{3.7}$$

$$F^{[4]}(x) \;=\; e\,\epsilon_{rstu}\,E^r\,\wedge\,E^s\,\wedge\,E^t\,\wedge\,E^u \quad ; \quad (e = \text{constant parameter}) \tag{3.8}$$

$$F^{[0]}_{\alpha\beta\gamma\delta}(x) \;=\; g_{\alpha\beta\gamma\delta} \;=\; \text{constant tensor} \tag{3.9}$$

we obtain an exact *bona fide* solution of the eleven-dimensional field equations of D=11 supergravity.

There are three possible 4-dimensional maximally symmetric Lorentzian manifolds

$$\mathcal{M}_4 = \begin{cases} \mathcal{M}_4 & \text{Minkowsky space} \\ \mathrm{dS}_4 & \text{de Sitter space} \\ \mathrm{AdS}_4 & \text{anti de Sitter space} \end{cases} \tag{3.10}$$

In any case Lorentz invariance imposes eqs. (3.5), (3.6), (3.7) while translation invariance imposes that the vacuum expectation value of the scalar fields $\Phi^{\alpha}{}_{\beta}(x)$ should be a constant matrix

$$< \Phi^{\alpha}{}_{\beta}(x) >= \mathcal{A}^{\alpha}_{\beta} \tag{3.11}$$

We are interested in 7-manifolds that preserve some residual supersymmetry in $D = 4$. This relates to the holonomy of M_7 which has to be restricted in order to allow for the existence of Killing spinors. In the next subsection we summarize those basic results from Kaluza Klein literature that are needed in our successive elaborations.

3.1 D=11 supergravity field equations and 7-manifolds of weak G_2 holonomy, i.e. Englert 7-manifolds

In order to admit at least one Killing spinor or more, the 7-manifold $\mathcal{M}_7$ necessarily must have a (weak) holonomy smaller than SO(7): at most G_2. The qualification weak refers to the definition of holonomy appropriate to compactifications on $\mathrm{AdS}_4 \times \mathcal{M}_7$ while the standard definition of holonomy is appropriate to compactifications on Ricci flat backgrounds $\mathrm{Mink}_4 \times \mathcal{M}_7$. To explain these concepts that were discovered in the eighties in contemporary language we have to recall the notion of G-structures. Indeed in the recent literature about flux compactifications the key geometrical notion exploited by most authors is precisely that of G-structures [25].

Following, for instance, the presentation of [25], if $\mathcal{M}_n$ is a differentiable manifold of dimension n, $T\mathcal{M}_n \xrightarrow{\pi} \mathcal{M}_n$ its tangent bundle and $F\mathcal{M}_n \xrightarrow{\pi} \mathcal{M}_n$ its frame bundle, we say that $\mathcal{M}_n$ admits a G-structure when the structural group of $F\mathcal{M}_n$ is reduced from the generic $\mathrm{GL(n,\mathbb{R})}$ to a proper subgroup $\mathrm{G} \subset \mathrm{GL(n,\mathbb{R})}$. Generically, tensors on $\mathcal{M}_n$ transform in representations of the structural group $\mathrm{GL(n,\mathbb{R})}$. If a G-structure reduces this latter to $\mathrm{G} \subset \mathrm{GL(n,\mathbb{R})}$, then the decomposition of an irreducible representation of $\mathrm{GL(n,\mathbb{R})}$, pertaining to a certain tensor t^p, with respect to the subgroup G may contain singlets. This means that on such a manifold $\mathcal{M}_n$ there may exist a certain tensor t^p which is G-invariant, and therefore globally defined. As recalled in [25] existence of a Riemannian metric g on $\mathcal{M}_n$ is equivalent to a reduction of the structural group $\mathrm{GL(n,\mathbb{R})}$ to O(n), namely to an O(n)-structure. Indeed, one can reduce the frame bundle by introducing orthonormal frames, the vielbein e^I, and, written in these frames, the metric is the O(n) invariant tensor δ_{IJ}. Similarly orientability corresponds to an SO(n)-structure and the existence of spinors on spin manifolds corresponds to a $\mathrm{Spin}(n)$-structure.

In the case of seven dimensions, an orientable Riemannian manifold $\mathcal{M}_7$, whose frame bundle has generically an SO(7) structural group admits a G_2-structure if and only if, in the basis provided by the orthonormal frames $\mathcal{B}^{\alpha}$, there exists an antisymmetric 3-tensor

$\phi_{\alpha\beta\gamma}$ satisfying the algebra of the octonionic structure constants:

$$\begin{aligned} \phi_{\alpha\beta\kappa}\,\phi_{\gamma\delta\kappa} &= \frac{1}{18}\,\delta^{\gamma\delta}_{\alpha\beta} - \frac{2}{3}\,\phi^\star_{\alpha\beta\gamma\delta} \\ -\frac{1}{6}\,\epsilon_{\kappa\rho\sigma\alpha\beta\gamma\delta}\,\phi^\star_{\alpha\beta\gamma\delta} &= \phi_{\kappa\rho\sigma} \end{aligned} \tag{3.12}$$

which is invariant, namely it is the same in all local trivializations of the SO(7) frame bundle. This corresponds to the algebraic definition of G_2 as that subgroup of SO(7) which acts as an automorphism group of the octonion algebra. Alternatively G_2 can be defined as the stability subgroup of the 8-dimensional spinor representation of SO(7). Hence we can equivalently state that a manifold $\mathcal{M}_7$ has a G_2-structure if there exists at least an invariant spinor η, which is the same in all local trivializations of the Spin(7) spinor bundle.

In terms of this invariant spinor the invariant 3-tensor $\phi_{\rho\sigma\kappa}$ has the form:

$$\phi^{\rho\sigma\kappa} = \frac{1}{6}\,\eta^T\,\tau^{\rho\sigma\kappa}\,\eta \tag{3.13}$$

and eq. (3.13) provides the relation between the two definitions of the G_2-structure.

On the other hand the manifold has not only a G_2-structure, but also G_2-holonomy if the invariant three-tensor $\phi_{\alpha\beta\kappa}$ is covariantly constant. Namely we must have:

$$0 = \nabla\phi^{\alpha\beta\gamma} \equiv d\phi^{\alpha\beta\gamma} + 3\,\mathcal{B}^{\kappa[\alpha}\,\phi^{\beta\gamma]\kappa} \tag{3.14}$$

where the 1-form $\mathcal{B}^{\alpha\beta}$ is the spin connection of $\mathcal{M}_7$. Alternatively the manifold has G_2-holonomy if the invariant spinor η is covariantly constant, namely if:

$$\exists\,\eta \in \Gamma(\mathrm{Spin}\mathcal{M}_7, \mathcal{M}_7) \quad \backslash \quad 0 = \nabla\,\eta \equiv d\eta - \frac{1}{4}\,\mathcal{B}^{\alpha\beta}\,\tau_{\alpha\beta}\,\eta \tag{3.15}$$

where τ^α $(\alpha = 1, \ldots, 7)$ are the 8×8 gamma matrices of the SO(7) Clifford algebra. The relation between the two definitions (3.14) and (3.15) of G_2-holonomy is the same as for the two definitions of the G_2-structure, namely it is given by eq. (3.13). As a consequence of its own definition a Riemannian 7-manifold with G_2 holonomy is Ricci flat. Indeed the integrability condition of eq. (3.15) yields:

$$\mathcal{R}^{\alpha\beta}{}_{\gamma\delta}\,\tau_{\alpha\beta}\,\eta = 0 \tag{3.16}$$

where $\mathcal{R}^{\alpha\beta}{}_{\gamma\delta}$ is the Riemann tensor of $\mathcal{M}_7$. From eq. (3.16), by means of a few simple algebraic manipulations one obtains two results:

- The curvature 2-form

  $$\mathcal{R}^{\alpha\beta} \equiv \mathcal{R}^{\alpha\beta}{}_{\gamma\delta}\,\mathcal{B}^\gamma \wedge \mathcal{B}^\delta \tag{3.17}$$

  is G_2 Lie algebra valued, namely it satisfies the condition:

  $$\phi^{\kappa\alpha\beta}\,\mathcal{R}^{\alpha\beta} = 0 \tag{3.18}$$

  which projects out the **7** of G_2 from the **21** of SO(7) and leaves with the adjoint **14**.

- The internal Ricci tensor is zero:

$$\mathcal{R}^{\alpha\kappa}{}_{\beta\kappa} = 0 \tag{3.19}$$

Next we consider the bosonic field equations of M-theory, namely the first and the last of eqs. (2.10). We make the compactification ansatz (3.1) where $\mathcal{M}_4$ is one of the three possibilities mentioned in eq. (3.10) and all of eqs. (3.5)–(3.9) hold true. Then we split the rigid index range as follows:

$$\underline{a}, \underline{b}, \underline{c}, \ldots = \begin{cases} \alpha, \beta, \gamma, \ldots &= 4, 5, 6, 7, 8, 9, 10 \;= \mathcal{M}_7 \text{ indices} \\ r, s, t, \ldots &= 0, 1, 2, 3 \qquad\quad = \mathcal{M}_4 \text{ indices} \end{cases} \tag{3.20}$$

and by following the conventions employed in [26] and using the results obtained in the same paper, we conclude that the compactification ansatz reduces the system of the first and last of (2.10) to the following one:

$$R^{rs}{}_{tu} = \lambda\, \delta^{rs}_{tu} \tag{3.21}$$

$$\mathcal{R}^{\alpha\kappa}{}_{\beta\kappa} = 3\,\nu\, \delta^{\alpha}_{\beta} \tag{3.22}$$

$$F_{rstu} = e\, \epsilon_{rstu} \tag{3.23}$$

$$g_{\alpha\beta\gamma\delta} = f\, \mathcal{F}_{\alpha\beta\gamma\delta} \tag{3.24}$$

$$\mathcal{F}^{\alpha\kappa\rho\sigma}\, \mathcal{F}_{\beta\kappa\rho\sigma} = \mu\, \delta^{\alpha}_{\beta} \tag{3.25}$$

$$\mathcal{D}^{\mu}\, \mathcal{F}_{\mu\kappa\rho\sigma} = \frac{1}{2}\, e\, \epsilon_{\kappa\rho\sigma\alpha\beta\gamma\delta}\, \mathcal{F}^{\alpha\beta\gamma\delta} \tag{3.26}$$

eq. (3.22) states that the internal manifold $\mathcal{M}_7$ must be an Einstein space. eqs. (3.23) and (3.24) state that there is a flux of the four-form both on 4-dimensional space-time $\mathcal{M}_4$ and on the internal manifold $\mathcal{M}_7$. The parameter e, which fixes the size of the flux on the four-dimensional space and was already introduced in eq. (3.8), is called the Freund-Rubin parameter [27]. As we are going to show, in the case that a non vanishing $\mathcal{F}^{\alpha\beta\gamma\delta}$ is required to exist, eqs. (3.25) and (3.26), are equivalent to the assertion that the manifold $\mathcal{M}_7$ has weak G_2 holonomy rather than G_2-holonomy, to state it in modern parlance [28]. In paper [29], manifolds admitting such a structure were instead named *Englert spaces* and the underlying notion of weak G_2 holonomy was already introduced there with the different name of *de Sitter* $\mathrm{SO}(7)^+$ *holonomy*.

Indeed eq. (3.26) which, in the language of the early eighties was named Englert equation [30] and which is nothing else but the first of equations (2.10), upon substitution of the Freund Rubin ansatz (3.23) for the external flux, can be recast in the following more revealing form: Let

$$\Phi^{\star} \equiv \mathcal{F}_{\alpha\beta\gamma\delta}\, \mathcal{B}^{\alpha} \wedge \mathcal{B}^{\beta} \wedge \mathcal{B}^{\gamma} \wedge \mathcal{B}^{\delta} \tag{3.27}$$

be a the constant 4-form on $\mathcal{M}_7$ defined by our non vanishing flux, and let

$$\Phi \equiv \frac{1}{24} \epsilon_{\alpha\beta\gamma\kappa\rho\sigma\tau}\, \mathcal{F}_{\kappa\rho\sigma\tau}\, \mathcal{B}^{\alpha} \wedge \mathcal{B}^{\beta} \wedge \mathcal{B}^{\gamma} \tag{3.28}$$

be its dual. Englert eq. (3.26) is just the same as writing:

$$\begin{aligned} d\Phi &= 12\, e\, \Phi^{\star} \\ d\Phi^{\star} &= 0 \end{aligned} \tag{3.29}$$

When the Freund Rubin parameter vanishes $e = 0$ we recognize in eq. (3.29) the statement that our internal manifold $\mathcal{M}_7$ has G_2-holonomy and hence it is Ricci flat. Indeed Φ is the G_2 invariant and covariantly constant form defining G_2-structure and G_2-holonomy. On the other hand the case $e \neq 0$ corresponds to the weak G_2 holonomy. Just as we reduced the existence of a closed three-form Φ to the existence of a G_2 covariantly constant spinor satisfying eq. (3.15) which allows to set the identification (3.13), in the same way eqs. (3.29) can be solved *if and only if* on $\mathcal{M}_7$ there exist a weak Killing spinor η satisfying the following defining condition:

$$\mathcal{D}_\alpha \eta = m\, e\, \tau_\alpha \eta \tag{3.30}$$

$$\Updownarrow$$

$$D\eta \equiv \left(d - \frac{1}{4}\mathcal{B}^{\alpha\beta}\tau_{\alpha\beta}\right)\eta = m\, e\, \mathcal{B}^\alpha \tau_\alpha \eta \tag{3.31}$$

where m is a numerical constant and e is the Freund-Rubin parameter, namely the only scale which at the end of the day will occur in the solution. The integrability of the above equation implies that the Ricci tensor be proportional to the identity, namely that the manifold is an Einstein manifold and furthermore fixes the proportionality constant:

$$\mathcal{R}^{\alpha\kappa}{}_{\beta\kappa} = 12\, m^2\, e^2\, \delta^\alpha_\beta \quad \longrightarrow \nu = 12\, m^2\, e^2 \tag{3.32}$$

In case such a spinor exists, by setting:

$$g_{\alpha\beta\gamma\delta} = \mathcal{F}_{\alpha\beta\gamma\delta} = \eta^T\, \tau_{\alpha\beta\gamma\delta}\eta \; = \; 24\, \phi^\star_{\alpha\beta\gamma\delta} \tag{3.33}$$

we find that Englert equation (3.26) is satisfied, provided we have:

$$m = -\frac{3}{2} \tag{3.34}$$

In this way Maxwell equation, namely the first of (2.10) is solved. Let us also note, as the authors of [29] did many years ago, that condition (3.30) can also be interpreted in the following way. The spin-connection $\mathcal{B}^{\alpha\beta}$ plus the vielbein $\mathcal{B}^\gamma$ define on any non Ricci flat 7-manifold $\mathcal{M}_7$ a connection which is actually SO(8) rather than SO(7) Lie algebra valued. In other words we have a principal SO(8) bundle which leads to an SO(8) spin bundle of which η is a covariantly constant section:

$$0 \; = \; \nabla^{\mathrm{SO}(8)}\eta = \left(\nabla^{\mathrm{SO}(7)} \; - \; m\, e\, \mathcal{B}^\alpha\, \tau_\alpha\right)\eta \tag{3.35}$$

The existence of η implies a reduction of the SO(8)-bundle. Indeed the stability subgroup of an SO(8) spinor is a well known subgroup $\mathrm{SO}(7)^+$ different from the standard SO(7) which, instead, stabilizes the vector representation. Hence the so named weak G_2 holonomy of the SO(7) spin connection $\mathcal{B}^{\alpha\beta}$ is the same thing as the $\mathrm{SO}(7)^+$ holonomy of the SO(8) Lie algebra valued *de Sitter connection* $\left\{\mathcal{B}^{\alpha\beta}, \mathcal{B}^\gamma\right\}$ introduced in [29] and normally discussed in the old literature on Kaluza Klein Supergravity.

We have solved Maxwell equation, but we still have to solve Einstein equation, namely the last of (2.10). To this effect we note that:

$$\mathcal{F}_{\beta\kappa\rho\sigma}\, \mathcal{F}^{\alpha\kappa\rho\sigma} \; = \; 24\, \delta^\alpha_\beta \quad \Longrightarrow \quad \mu = 24 \tag{3.36}$$

and we observe that Einstein equation reduces to the following two conditions on the parameters (see [26] for details):

$$\begin{aligned} \frac{3}{2}\,\lambda &= -\left(24\,e^2 + \frac{7}{2}\,\mu\,f^2\right) \\ 3\,\nu &= 12\,e^2 + \frac{5}{2}\,\mu\,f^2 \end{aligned} \tag{3.37}$$

¿From eqs. (3.37) we conclude that there are only three possible kind of solutions.

a The flat solutions of type

$$\mathcal{M}_{11} \;=\; \mathrm{Mink}_4 \otimes \underbrace{\mathcal{M}_7}_{\text{Ricci flat}} \tag{3.38}$$

where both $D = 4$ space-time and the internal 7-space are Ricci flat. These compactifications correspond to $e = 0$ and $F_{\alpha\beta\gamma\delta} = 0 \Rightarrow g_{\alpha\beta\gamma\delta} = 0$.

b The Freund Rubin solutions of type

$$\mathcal{M}_{11} \;=\; \mathrm{AdS}_4 \otimes \underbrace{\mathcal{M}_7}_{\text{Einst. manif.}} \tag{3.39}$$

These correspond to anti de Sitter space in 4-dimensions, whose radius is fixed by the Freund Rubin parameter $e \neq 0$ times any Einstein manifold in 7-dimensions with no internal flux, namely $g_{\alpha\beta\gamma\delta} = 0$. In this case from eq. (3.37) we uniquely obtain:

$$R^{rs}{}_{tu} \;=\; -16\,e^2\,\delta^{rs}_{tu} \tag{3.40}$$

$$\mathcal{R}^{\alpha\kappa}_{\beta\kappa} \;=\; 12\,e^2\,\delta^{\alpha}_{\beta} \tag{3.41}$$

$$F_{rstu} \;=\; e\,\epsilon_{rstu} \tag{3.42}$$

$$F_{\alpha\beta\gamma\delta} \;=\; 0 \tag{3.43}$$

c The Englert type solutions

$$\mathcal{M}_{11} \;=\; \mathrm{AdS}_4 \otimes \underbrace{\mathcal{M}_7}_{\substack{\text{Einst. manif.} \\ \text{weak } \mathrm{G}_2 \text{ hol}}} \tag{3.44}$$

These correspond to anti de Sitter space in 4-dimensions ($e \neq 0$) times a 7-dimensional Einstein manifold which is necessarily of weak G_2 holonomy in order to support a consistent non vanishing internal flux $g_{\alpha\beta\gamma\delta}$. In this case combining eqs. (3.37) with the previous ones we uniquely obtain:

$$\lambda = -30\,e^2 \quad ; \quad f = \pm\frac{1}{2}\,e \tag{3.45}$$

As we already mentioned in the introduction there exist several compact manifolds of weak G_2 holonomy. In particular all the coset manifolds $\mathcal{G}/\mathcal{H}$ of weak G_2 holonomy were classified and studied in the Kaluza Klein supergravity age [31, 26, 32 – 36, 29, 37, 38] and they were extensively reconsidered in the context of the AdS/CFT correspondence [39 – 43].

In the present paper we study the supergauge completion of compactifications of the Freund Rubin type, namely on eleven-manifolds of the form:

$$\mathcal{M}_{11} = \mathrm{AdS}_4 \times \frac{\mathcal{G}}{\mathcal{H}} \tag{3.46}$$

with no internal flux $g_{\alpha\beta\gamma\delta}$ switched on. As it was extensively explained in [44] and further developed in [39 – 43], if the compact coset $\mathcal{G}/\mathcal{H}$ admits $\mathcal{N} \le 8$ Killing spinors η_A, namely $N \le 8$ independent solutions of equation (3.30) with $m = 1$, then the isometry group $\mathcal{G}$ is necessarily of the form:

$$\mathcal{G} = \mathrm{SO}(\mathcal{N}) \times \mathrm{G}_{flavor} \tag{3.47}$$

where G_{flavor} is some appropriate Lie group. In this case the isometry supergroup of the considered D=11 supergravity background is:

$$\mathrm{Osp}(\mathcal{N}\,|\,4) \times \mathrm{G}_{flavor} \tag{3.48}$$

and the spectrum of fluctuations of the background arranges into $\mathrm{Osp}(\mathcal{N}\,|\,4)$ supermultiplets furthermore assigned to suitable representations of the bosonic flavor group.

4. The SO(8) spinor bundle and the holonomy tensor

We come next to discuss a very important property of 7-manifolds with a spin structure which plays a crucial role in understanding the supergauge completion. This is the existence of an SO(8) vector bundle whose non trivial connection is defined by the riemannian structure of the manifold. To introduce this point and in order to illustrate its relevance to our problem we begin by considering a basis of $D = 11$ gamma matrices well adapted to the compactification on $\mathrm{AdS}_4 \times \mathcal{M}_7$.

4.1 The well adapted basis of gamma matrices

According to the tensor product representation well adapted to the compactification, the $D = 11$ gamma matrices can be written as follows:

$$\begin{aligned} \Gamma_a &= \gamma_a \otimes \mathbf{1}_{8\times 8} \quad (a = 0,1,2,3) \\ \Gamma_{3+\alpha} &= \gamma_5 \otimes \tau_\alpha \quad (\alpha = 1,\dots,7) \end{aligned} \tag{4.1}$$

where, following [4] and the old Kaluza Klein supergravity literature [29, 44, 33] the matrices τ_α are the real antisymmetric realization of the SO(7) Clifford algebra with negative metric:

$$\{\tau_\alpha\,,\,\tau_\beta\} = -\,2\,\delta_{\alpha\beta} \quad ; \quad \tau_\alpha = -\,(\tau_\alpha)^T \tag{4.2}$$

In this basis the charge conjugation matrix is given by:

$$C = \mathcal{C} \otimes \mathbf{1}_{8\times 8} \tag{4.3}$$

where $\mathcal{C}$ is the charge conjugation matrix in $d = 4$:

$$\mathcal{C}\,\gamma_a\,\mathcal{C}^{-1} = -\gamma_a^T \quad ; \quad \mathcal{C}^T = -\mathcal{C} \tag{4.4}$$

4.2 The $\mathfrak{so}(8)$-connection and the holonomy tensor

Next we observe that using these matrices the covariant derivative introduced in equation (3.35) defines a universal $\mathfrak{so}(8)$-connection on the spinor bundle which is given once the riemannian structure, namely the vielbein and the spin connection are given $\{\mathcal{B}^\alpha, B^{\alpha\beta}\}$:

$$\mathbf{U}^{\mathfrak{so}(8)} \equiv -\frac{1}{4}\,\mathcal{B}^{\alpha\beta}\,\tau_{\alpha\beta} - e\,\mathcal{B}^\alpha\,\tau_\alpha \tag{4.5}$$

More precisely and following the index conventions presented in appendix A, let $\zeta_{\underline{A}}$ be an orthonormal basis:

$$\overline{\zeta}_A\,\zeta_B = \delta_{AB} \tag{4.6}$$

of sections of the spinor bundle over the Einstein manifold M_7. Any spinor can be written as a linear combination of these sections that are real. Furthermore the bar operation in this case is simply the transposition. Hence, if we consider the $\mathfrak{so}(8)$ covariant derivative of any of these sections, this is a spinor and, as such, it can be expressed as a linear combinations of the same:

$$\nabla^{\mathfrak{so}(8)}\,\zeta_A \equiv \left(d + \mathbf{U}^{\mathfrak{so}(8)}\right)\zeta_A = \mathbf{U}_{AB}\,\zeta_{\mathrm{B}} \tag{4.7}$$

According to standard lore the 1-form valued, antisymmetric 8×8 matrix $\mathbf{U}_{AB}$ defined by eq. (4.7) is the $\mathfrak{so}(8)$-connection in the chosen basis of sections. If the manifold $\mathcal{M}_7$ admits $\mathcal{N}$ Killing spinors, then it follows that we can choose an orthonormal basis where the first $\mathcal{N}$ sections are Killing spinors:

$$\zeta_{\underline{A}} = \eta_{\underline{A}} \quad ; \ \nabla^{\mathfrak{so}(8)}\,\eta_{\underline{A}} = 0 \quad , \quad \underline{A} = 1, \dots, \mathcal{N} \tag{4.8}$$

and the remaining $8-\mathcal{N}$ elements of the basis, whose covariant derivative does not vanish are orthogonal to the Killing spinors:

$$\begin{aligned} \zeta_\Lambda &= \xi_{\overline{A}} \quad ; \ \nabla^{\mathfrak{so}(8)}\,\xi_{\overline{A}} \neq 0 \quad , \quad \overline{A} = 1, \dots, 8-\mathcal{N} \\ \overline{\xi}_{\overline{B}}\,\eta_{\underline{A}} &= 0 \\ \overline{\xi}_{\overline{B}}\,\xi_{\overline{C}} &= \delta_{\overline{BC}} \end{aligned} \tag{4.9}$$

It is then evident from eqs. (4.8) and (4.9) that the $\mathfrak{so}(8)$-connection $\mathbf{U}_{AB}$ takes values only in a subalgebra $\mathfrak{so}(8-\mathcal{N}) \subset \mathfrak{so}(8)$ and has the following block diagonal form:

$$\mathbf{U}_{AB} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{U}_{\overline{AB}} \end{array}\right) \tag{4.10}$$

Squaring the SO(8)-covariant derivative, we find

$$\begin{aligned} \nabla^2\,\zeta_A &= \underbrace{\left(d\mathbf{U}_{AB} - \mathbf{U}_{\underline{AC}} \wedge \mathbf{U}_{CB}\right)}_{\mathcal{F}_{AB}[\mathbf{U}]}\,\zeta_{\underline{B}} \\ &= -\frac{1}{4}\underbrace{\left(\mathcal{R}^{\gamma\delta}{}_{\alpha\beta} - 4\,e^2\,\delta^{\gamma\delta}{}_{\alpha\beta}\right)}_{\mathcal{C}^{\gamma\delta}{}_{\alpha\beta}}\,\tau_{\gamma\delta}\,\zeta_{\underline{A}} \end{aligned} \tag{4.11}$$

where $\mathcal{C}^{\gamma\delta}{}_{\alpha\beta}$ is the so called holonomy tensor, essentially identical with the Weyl tensor of the considered Einstein 7-manifold.

4.3 The holonomy tensor and superspace

As a further preparation to our subsequent discussion of the gauge completion let us now consider the form taken on the $\mathrm{AdS}_4 \times \mathcal{G}/\mathcal{H}$ backgrounds by the operator $K_{\underline{ab}}$ introduced in equation (2.19) and governing the mechanism of supersymmetry breaking. We will see that it is just simply related to the holonomy tensor discussed in the previous section, namely to the field strength of the SO(8)-connection on the spinor bundle. To begin with we calculate the operator $F_{\underline{a}}$ introduced in eqs. (2.13), (2.15). Explicitly using the well adapted basis (4.1) for gamma matrices we find:

$$F_{\underline{a}} = \begin{cases} F_a \;=\; -2\, e\, \gamma_a\, \gamma_5 \otimes \mathbf{1}_8 \\ F_\alpha \;=\; -e\, \mathbf{1}_4 \otimes \tau_\alpha \end{cases} \tag{4.12}$$

Using this input we obtain:

$$K_{\underline{ab}} = \begin{cases} K_{ab} \;=\; 0 \\ K_{a\beta} \;=\; 0 \\ K_{\alpha\beta} \;=\; -\frac{1}{4}\, \underbrace{\left(\mathcal{R}^{\gamma\delta}{}_{\alpha\beta} - 4\, e^2\, \delta^{\gamma\delta}{}_{\alpha\beta}\right)}_{C^{\gamma\delta}{}_{\alpha\beta}}\, \tau_{\gamma\delta} \end{cases} \tag{4.13}$$

Where the tensor $C^{\gamma\delta}{}_{\alpha\beta}$ defined by the above equation is named the holonomy tensor and it is an intrinsic geometric property of the compact internal manifold $\mathcal{M}_7$. As we see the holonomy tensor vanishes only in the case of $\mathcal{M}_7 = \mathcal{S}^7$ when the Riemann tensor is proportional to an antisymmetrized Kronecker delta, namely, when the internal Einstein 7-manifold is maximally symmetric. The holonomy tensor is a 21×21 matrix which projects the SO(7) Lie algebra to a subalgebra:

$$\mathbb{H}_{hol} \subset \mathrm{SO}(7) \tag{4.14}$$

with respect to which the 8-component spinor representation should contain singlets in order for unbroken supersymmetries to survive. Indeed the holonomy tensor appears in the integrability condition for Killing spinors. Indeed squaring the defining equation of Killing spinors with $m = 1$ we get the consistency condition:

$$C^{\gamma\delta}{}_{\alpha\beta}\, \tau_{\gamma\delta}\, \eta \;=\; 0 \tag{4.15}$$

which states that the Killing spinor directions are in the kernel of the operators $C^{\gamma\delta}{}_{\alpha\beta}\, \tau_{\gamma\delta}$, namely are singlets of the subalgebra $\mathbb{H}_{hol}$ generated by them.

In view of this we conclude that the gravitino field strength has the following structure:

$$\rho_{\underline{ab}} = \begin{cases} \rho_{ab} \;=\; 0 \\ \rho_{a\beta} \;=\; 0 \\ \rho_{\alpha\beta} \;\neq\; 0\,; \begin{cases} \text{zero at } \theta = 0 \\ \text{depends only on the broken } \theta\text{.s} \end{cases} \end{cases} \tag{4.16}$$

As a preparation for our next coming discussion it is now useful to remind the reader that the list of homogeneous 7-manifolds $\mathcal{G}/\mathcal{H}$ of Englert type which preserve at least two

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$\mathcal{N}$	Name	Coset	Holon. $\mathfrak{so}(8)$ bundle	Fibration
8	$\mathbb{S}^7$	$\frac{\mathrm{SO}(8)}{\mathrm{SO}(7)}$	1	$\begin{cases} \mathbb{S}^7 \stackrel{\pi}{\Longrightarrow} \mathbb{P}^3 \\ \forall p \in \mathbb{P}^3 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	M^{111}	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{SU}(2)\times\mathrm{U}(1)\times\mathrm{U}(1)}$	SU(3)	$\begin{cases} M^{111} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^2 \times \mathbb{P}^1 \\ \forall p \in \mathbb{P}^2 \times \mathbb{P}^1 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	Q^{111}	$\frac{\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)\times\mathrm{U}(1)}$	SU(3)	$\begin{cases} Q^{111} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \\ \forall p \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
2	$V^{5,2}$	$\frac{\mathrm{SO}(5)}{\mathrm{SO}(2)}$	SU(3)	$\begin{cases} V^{5,2} \stackrel{\pi}{\Longrightarrow} M_a \sim \text{quadric in } \mathbb{P}^4 \\ \forall p \in M_a ; \pi^{-1}(p) \sim \mathbb{S}^1 \end{cases}$
3	N^{010}	$\frac{\mathrm{SU}(3)\times\mathrm{SU}(2)}{\mathrm{SU}(2)\times\mathrm{U}(1)}$	SU(2)	$\begin{cases} N^{010} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^2 \\ \forall p \in \mathbb{P}^2 ; \pi^{-1}(p) \sim \mathbb{S}^3 \end{cases}$

Table 1: The homogeneous 7-manifolds that admit at least 2 Killing spinors are all sasakian or tri-sasakian. This is evident from the fibration structure of the 7-manifold, which is either a fibration in circles $\mathbb{S}^1$ for the $\mathcal{N} = 2$ cases or a fibration in $\mathbb{S}^3$ for the unique $\mathcal{N} = 3$ case corresponding to the N^{010} manifold

$\mathcal{N}$	Name	Coset	Holon. $\mathfrak{so}(8)$ bundle
1	$\mathbb{S}^7_{squashed}$	$\frac{\mathrm{SO}(5)\times\mathrm{SO}(3)}{\mathrm{SO}(3)\times\mathrm{SO}(3)}$	$\mathrm{SO}(7)^+$
1	$\mathrm{N}^{\mathrm{pqr}}$	$\frac{\mathrm{SU}(3)\times\mathrm{U}(1)}{\mathrm{U}(1)\times\mathrm{U}(1)}$	$\mathrm{SO}(7)^+$

Table 2: The homogeneous 7-manifolds that admit just one Killing spinors are the squashed 7-sphere and the infinite family of $\mathrm{N}^{\mathrm{pqr}}$ manifolds for $pqr \neq 010$.

supersymmetries ($\mathcal{N} \geq 2$) is extremely short. It consists of the sasakian or tri-sasakian homogeneous manifolds which are displayed in table 1. For these cases our strategy in order to obtain the supergauge completion will be based on a superextension of the sasakian fibration. The cases with $\mathcal{N} = 1$ are somewhat more involved since such a weapon is not in our stoke. These cases are also ultra-few and they are displayed in table 2.

5. The OSp($\mathcal{N}$|4) supergroup, its superalgebra and its supercosets

The key ingredients in the construction of the supergauge completion of $\mathrm{AdS}_4 \times \mathcal{G}/\mathcal{H}$ are provided by supercoset manifolds of the supergroup OSp($\mathcal{N}$|4) [23, 45, 46, 35, 36]. For this reason we dedicate this section to an in depth analysis of such a supergroup to the structure of its superalgebra described by appropriate Maurer Cartan equations and to the explicit construction of coset representatives for relevant instances of supercosets of the form $\mathrm{OSp}(\mathcal{N}|4)/H$. This lore will be crucial in our subsequent discussions.

5.1 The superalgebra

The real form $\mathfrak{osp}(\mathcal{N}|4)$ of the complex $\mathfrak{osp}(\mathcal{N}|4,\mathbb{C})$ Lie superalgebra which is relevant for the study of $\mathrm{AdS}_4 \times \mathcal{G}/\mathcal{H}$ compactifications is that one where the ordinary Lie subalgebra is the following:

$$\mathfrak{sp}(4,\mathbb{R}) \times \mathfrak{so}(\mathcal{N}) \subset \mathfrak{osp}(\mathcal{N}|4) \tag{5.1}$$

This is quite obvious because of the isomorphism $\mathfrak{sp}(4,\mathbb{R}) \simeq \mathfrak{so}(2,3)$ which identifies $\mathfrak{sp}(4,\mathbb{R})$ with the isometry algebra of anti de Sitter space. The compact algebra $\mathfrak{so}(8)$ is instead the R-symmetry algebra acting on the supersymmetry charges.

The superalgebra $\mathfrak{osp}(\mathcal{N}|4)$ can be introduced as follows: consider the two graded $(4+\mathcal{N}) \times (4+\mathcal{N})$ matrices:

$$\widehat{C} = \left(\begin{array}{c|c} C\,\gamma_5 & 0 \\ \hline 0 & -\frac{\mathrm{i}}{4\,e}\,\mathbf{1}_{\mathcal{N}\times\mathcal{N}} \end{array}\right) \; ; \; \widehat{H} = \left(\begin{array}{c|c} \mathrm{i}\,\gamma_0\,\gamma_5 & 0 \\ \hline 0 & -\frac{1}{4\,e}\,\mathbf{1}_{\mathcal{N}\times\mathcal{N}} \end{array}\right) \tag{5.2}$$

where C is the charge conjugation matrix in $D = 4$. The matrix $\widehat{C}$ has the property that its upper block is antisymmetric while its lower one is symmetric. On the other hand, the matrix H has the property that both its upper and lower blocks are hermitian. The $\mathfrak{osp}(\mathcal{N}|4)$ Lie algebra is then defined as the set of graded matrices Λ satisfying the two conditions:

$$\Lambda^T\,\widehat{C} + \widehat{C}\,\Lambda = 0 \tag{5.3}$$

$$\Lambda^\dagger\,\widehat{H} + \widehat{H}\,\Lambda = 0 \tag{5.4}$$

eq. (5.3) defines the complex $\mathrm{osp}(\mathcal{N}|4)$ superalgebra while eq. (5.4) restricts it to the appropriate real section where the ordinary Lie subalgebra is (5.1). The specific form of the matrices $\widehat{C}$ and $\widehat{H}$ is chosen in such a way that the complete solution of the constraints (5.3), (5.4) takes the following form:

$$\Lambda = \left(\begin{array}{c|c} -\frac{1}{4}\,\omega^{ab}\,\gamma_{ab} - 2\,e\,\gamma_a\,\gamma_5\,E^a & \psi_A \\ \hline 4\,\mathrm{i}\,e\,\overline{\psi}_B\,\gamma_5 & -\,e\,\mathcal{A}_{AB} \end{array}\right) \tag{5.5}$$

and the Maurer-Cartan equations

$$d\,\Lambda + \Lambda \wedge \Lambda = 0 \tag{5.6}$$

read as follows:

$$\begin{aligned} d\omega^{ab} - \omega^{ac} \wedge \omega^{db}\,\eta_{cd} + 16e^2 E^a \wedge E^b &= -\mathrm{i}\,2e\,\overline{\psi}_A \wedge \gamma^{ab}\gamma^5\psi_A, \\ dE^a - \omega^a{}_c \wedge E^c &= \mathrm{i}\frac{1}{2}\,\overline{\psi}_A \wedge \gamma^a\psi_A, \\ d\psi_A - \frac{1}{4}\omega^{ab} \wedge \gamma_{ab}\psi_A - e\mathcal{A}_{AB} \wedge \psi_B &= 2e\,E^a \wedge \gamma_a\gamma_5\psi_A, \\ d\mathcal{A}_{AB} - e\mathcal{A}_{AC} \wedge \mathcal{A}_{CB} &= 4\,\mathrm{i}\overline{\psi}_A \wedge \gamma_5\psi_B\,. \end{aligned} \tag{5.7}$$

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Interpreting E^a as the vielbein, ω^{ab} as the spin connection, and ψ^a as the gravitino 1-form, eqs. (5.7) can be viewed as the structural equations of a supermanifold $\mathrm{AdS}_{4|\mathcal{N}\times 4}$ extending anti de Sitter space with $\mathcal{N}$ Majorana supersymmetries. Indeed the gravitino 1-form is a Majorana spinor since, by construction, it satisfies the reality condition

$$C\,\overline{\psi}_A^T = \psi_A\,, \qquad \overline{\psi}_A \equiv \psi_A^\dagger\,\gamma_0\,. \tag{5.8}$$

The supermanifold $\mathrm{AdS}_{4|\mathcal{N}\times 4}$ can be identified with the following supercoset:

$$\mathcal{M}^{4|4\mathcal{N}}_{osp} \equiv \frac{\mathrm{Osp}(\mathcal{N}\,|\,4)}{\mathrm{SO}(\mathcal{N})\times\mathrm{SO}(1,3)} \tag{5.9}$$

Alternatively, the Maurer Cartan equations can be written in the following more compact form:

$$\begin{aligned} d\Delta^{xy} + \Delta^{xz}\wedge\Delta^{ty}\,\epsilon_{zt} &= -\,4\,\mathrm{i}\,e\,\Phi^x_A\wedge\Phi^y_A, \\ d\mathcal{A}_{AB} - e\mathcal{A}_{AC}\wedge\mathcal{A}_{CB} &= 4\,\mathrm{i}\Phi^x_A\wedge\Phi^y_B\,\epsilon_{xy} \\ d\Phi^x_A + \Delta^{xy}\wedge\epsilon_{yz}\,\Phi^z_A - e\,\mathcal{A}_{AB}\wedge\Phi^x_B &= 0 \end{aligned} \tag{5.10}$$

where all 1-forms are real and, according to the conventions discussed in appendix A, the indices x, y, z, t are symplectic and take four values. The real symmetric bosonic 1-form $\Omega^{xy} = \Omega^{yx}$ encodes the generators of the Lie subalgebra $\mathfrak{sp}(4,\mathbb{R})$, while the antisymmetric real bosonic 1-form $\mathcal{A}_{AB} = -\mathcal{A}_{BA}$ encodes the generators of the Lie subalgebra $\mathfrak{so}(\mathcal{N})$. The fermionic 1-forms Φ^x_A are real and, as indicated by their indices, they transform in the fundamental 4-dim representation of $\mathfrak{sp}(4,\mathbb{R})$ and in the fundamental $\mathcal{N}$-dim representation of $\mathfrak{so}(\mathcal{N})$. Finally,

$$\epsilon_{xy} = -\epsilon_{yx} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \tag{5.11}$$

is the symplectic invariant metric.

The relation between the formulation (5.7) and (5.10) of the same Maurer Cartan equations is provided by the Majorana basis of $d = 4$ gamma matrices discussed in appendix B.2. Using eq. (B.8), the generators γ_{ab} and $\gamma_a\,\gamma_5$ of the anti de Sitter group SO(2, 3) turn out to be all given by real symplectic matrices, as is explicitly shown in eq. (B.10) and the matrix $\mathcal{C}\,\gamma_5$ turns out to be proportional to ϵ_{xy} as shown in eq. (B.9). On the other hand a Majorana spinor in this basis is proportional to a real object times a phase factor $\exp[-\,\pi\,\mathrm{i}\,/\,4]$.

Hence eqs. (5.7) and eqs. (5.10) are turned ones into the others upon the identifications:

$$\begin{aligned} \Omega^{xy}\,\epsilon_{yz} \equiv \Omega^x{}_z &\leftrightarrow -\tfrac{1}{4}\,\omega^{ab}\,\gamma_{ab} - 2\,e\,\gamma_a\,\gamma_5\,E^a \\ \mathcal{A}_{AB} &\leftrightarrow \mathcal{A}_{AB} \\ \psi^x_A &\leftrightarrow \exp\left[\tfrac{-\pi\mathrm{i}}{4}\right]\,\Phi^x_A \end{aligned} \tag{5.12}$$

As is always the case, the Maurer Cartan equations are just a property of the (super) Lie algebra and hold true independently of the (super) manifold on which the 1-forms are realized: on the supergroup manifold or on different supercosets of the same supergroup.

5.2 The relevant supercosets and their relation

We have already introduced the supercoset (5.9) which includes anti de Sitter space and has 4 bosonic coordinates and $4 \times \mathcal{N}$ fermionic ones. Let us also consider the following pure fermionic coset:

$$\mathcal{M}^{0|4\mathcal{N}}_{osp} = \frac{\mathrm{Osp}(\mathcal{N}\,|\,4)}{\mathrm{SO}(\mathcal{N}) \times \mathrm{Sp}(4,\mathbb{R})} \tag{5.13}$$

There is an obvious relation between these two supercosets that can be formulated in the following way:

$$\mathcal{M}^{4|4\mathcal{N}}_{osp} \sim \mathrm{AdS}_4 \times \mathcal{M}^{0|4\mathcal{N}}_{osp} \tag{5.14}$$

In order to explain the actual meaning of eq. (5.14) we proceed as follows. Let the graded matrix $\mathbb{L} \in \mathrm{Osp}(\mathcal{N}|4)$ be the coset representative of the coset $\mathcal{M}^{4|4\mathcal{N}}_{osp}$, such that the Maurer Cartan form Λ of eq. (5.5) can be identified as:

$$\Lambda = \mathbb{L}^{-1}\, d\mathbb{L} \tag{5.15}$$

Let us now factorize $\mathbb{L}$ as follows:

$$\mathbb{L} = \mathbb{L}_F\, \mathbb{L}_B \tag{5.16}$$

where $\mathbb{L}_F$ is a coset representative for the coset:

$$\frac{\mathrm{Osp}(\mathcal{N}\,|\,4)}{\mathrm{SO}(\mathcal{N}) \times \mathrm{Sp}(4,\mathbb{R})} \ni \mathbb{L}_F \tag{5.17}$$

and $\mathbb{L}_B$ is the $\mathrm{Osp}(\mathcal{N}|4)$ embedding of a coset representative of AdS_4, namely:

$$\mathbb{L}_B = \left(\begin{array}{c|c} \mathrm{L_B} & 0 \\ \hline 0 & \mathbf{1}_{\mathcal{N}} \end{array}\right) \quad ; \quad \frac{\mathrm{Sp}(4,\mathbb{R})}{\mathrm{SO}(1,3)} \ni \mathrm{L_B} \tag{5.18}$$

In this way we find:

$$\Lambda = \mathbb{L}_B^{-1}\, \Lambda_F\, \mathbb{L}_B + \mathbb{L}_B^{-1}\, d\,\mathbb{L}_B \tag{5.19}$$

Let us now write the explicit form of Λ_F in analogy to eq. (5.5):

$$\Lambda_F = \left(\begin{array}{c|c} \Delta_F & \Theta_A \\ \hline 4\,\mathrm{i}\, e\, \overline{\Theta}_A\, \gamma_5 & -\, e\, \widetilde{\mathcal{A}}_{AB} \end{array}\right) \tag{5.20}$$

where Θ_A is a Majorana-spinor valued fermionic 1-form and where Δ_F is an $\mathfrak{sp}(4,\mathbb{R})$ Lie algebra valued 1-form presented as a 4×4 matrix. Both Θ_A as Δ_F and $\widetilde{\mathcal{A}}_{AB}$ depend only on the fermionic θ coordinates and differentials.

On the other hand we have:

$$\mathbb{L}_B^{-1}\, d\,\mathbb{L}_B = \left(\begin{array}{c|c} \Delta_B & 0 \\ \hline 0 & 0 \end{array}\right) \tag{5.21}$$

where the Ω_B is also an $\mathfrak{sp}(4,\mathbb{R})$ Lie algebra valued 1-form presented as a 4×4 matrix, but it depends only on the bosonic coordinates x^μ of the anti de Sitter space AdS_4. Indeed, according to eq. (5.5) we can write:

$$\Delta_B \;=\; -\frac{1}{4}\,B^{ab}\,\gamma_{ab} \;-\; 2\,e\,\gamma_a\,\gamma_5\,B^a \tag{5.22}$$

where $\left\{B^{ab}\,,\,B^a\right\}$ are respectively the spin-connection and the vielbein of AdS_4, just as $\left\{\mathcal{B}^{\alpha\beta}\,,\,\mathcal{B}^\alpha\right\}$ are the connection and vielbein of the internal coset manifold $\mathcal{M}_7$.

Inserting now these results into eq. (5.19) and comparing with eq. (5.5) we obtain:

$$\begin{aligned}\psi_A &= \mathrm{L}_\mathrm{B}^{-1}\,\Theta_A\\ \mathcal{A}_{AB} &= \widetilde{\mathcal{A}}_{AB}\\ -\frac{1}{4}\,\omega^{ab}\,\gamma_{ab} \;-\; 2\,e\,\gamma_a\,\gamma_5\,E^a &= -\frac{1}{4}\,B^{ab}\,\gamma_{ab} \;-\; 2\,e\,\gamma_a\,\gamma_5\,B^a \;+\; \mathrm{L}_\mathrm{B}^{-1}\,\Delta_F\,\mathrm{L}_\mathrm{B}\end{aligned} \tag{5.23}$$

The above formulae encode an important information. They show how the supervielbein and the superconnection of the supermanifold (5.9) can be constructed starting from the vielbein and connection of AdS_4 space plus the Maurer Cartan forms of the purely fermionic supercoset (5.13). In other words formulae (5.23) provide the concrete interpretation of the direct product (5.14). This will also be our starting point for the actual construction of the supergauge completion in the case of maximal supersymmetry and for its generalization to the cases of less supersymmetry.

5.3 Finite supergroup elements

We studied the $\mathfrak{osp}(\mathcal{N}|4)$ superalgebra but for our purposes we cannot confine ourselves to the superalgebra, we need also to consider finite elements of the corresponding supergroup. In particular the supercoset representative. Elements of the supergroup are described by graded matrices of the form:

$$M = \left(\begin{array}{c|c} A & \Theta \\ \hline \Pi & D\end{array}\right) \tag{5.24}$$

where A, D are submatrices made out of even elements of a Grassmann algebra while Θ, Π are submatrices made out of odd elements of the same Grassmann algebra. It is important to recall, that the operations of transposition and hermitian conjugation are defined as follows on graded matrices:

$$\begin{aligned}M^T &= \left(\begin{array}{c|c} A^T & \Pi^T \\ \hline -\,\Theta^T & D^T\end{array}\right)\\ M^\dagger &= \left(\begin{array}{c|c} A^\dagger & \Pi^\dagger \\ \hline \Theta^\dagger & D^\dagger\end{array}\right)\end{aligned} \tag{5.25}$$

This is done in order to preserve for the supertrace the same formal properties enjoyed by the trace of ordinary matrices:

$$\begin{aligned}\mathrm{Str}\;(M) &= \mathrm{Tr}\;(A) - \mathrm{Tr}\;(D)\\ \mathrm{Str}\;(M_1\,M_2) &= \mathrm{Str}\;(M_2\,M_1)\end{aligned} \tag{5.26}$$

eqs. (5.25) and (5.26) have an important consequence. The consistency of the equation:

$$M^\dagger = \left(M^T\right)^\star \tag{5.27}$$

implies that the complex conjugate operation on a super matrix must be defined as follows:

$$M^\star = \left(\begin{array}{c|c} A^\star & -\Theta^\star \\ \hline \Pi^\star & D^\star \end{array}\right) \tag{5.28}$$

Let us now observe that in the Majorana basis which we have adopted we have:

$$\begin{aligned} \widehat{C} &= \mathrm{i}\,\left(\begin{array}{c|c} \epsilon & 0 \\ \hline 0 & -\frac{1}{4e}\,\mathbf{1}_{\mathcal{N}\times\mathcal{N}} \end{array}\right) = \mathrm{i}\,\widehat{\epsilon} \\ \widehat{H} &= \left(\begin{array}{c|c} \mathrm{i}\,\epsilon & 0 \\ \hline 0 & -\frac{1}{4e}\,\mathbf{1}_{\mathcal{N}\times\mathcal{N}} \end{array}\right) \end{aligned} \tag{5.29}$$

where the 4×4 matrix ϵ is given by eq. (B.9). Therefore in this basis an orthosymplectic group element $\mathbb{L} \in \mathrm{OSp}(\mathcal{N}|4)$ which satisfies:

$$\mathbb{L}^T\,\widehat{C}\,\mathbb{L} = \widehat{C} \tag{5.30}$$

$$\mathbb{L}^\dagger\,\widehat{H}\,\mathbb{L} = \widehat{H} \tag{5.31}$$

has the following structure:

$$\mathbb{L} = \left(\begin{array}{c|c} \mathcal{S} & \exp\left[-\mathrm{i}\frac{\pi}{4}\right]\Theta \\ \hline \exp\left[-\mathrm{i}\frac{\pi}{4}\right]\Pi & \mathcal{O} \end{array}\right) \tag{5.32}$$

where the bosonic sub-blocks $\mathcal{S}, \mathcal{O}$ are respectively 4×4 and $\mathcal{N} \times \mathcal{N}$ and real, while the fermionic ones Θ, Π are respectively $4 \times \mathcal{N}$ and $\mathcal{N} \times 4$ and also real.

The orthosymplectic conditions (5.30) translate into the following conditions on the sub-blocks:

$$\begin{aligned} \mathcal{S}^T\,\epsilon\,\mathcal{S} &= \epsilon - \mathrm{i}\,\frac{1}{4e}\,\Pi^T\,\Pi \\ \mathcal{O}^T\,\mathcal{O} &= \mathbf{1} + \mathrm{i}\,4e\,\Theta^T\,\epsilon\,\Theta \\ \mathcal{S}^T\,\epsilon\,\Theta &= -\frac{1}{4e}\,\Pi^T\,\mathcal{O} \end{aligned} \tag{5.33}$$

As we see, when the fermionic off-diagonal sub-blocks are zero the diagonal ones are respectively a symplectic and an orthogonal matrix.

If the graded matrix $\mathbb{L}$ is regarded as the coset representative of either one of the two supercosets (5.9), (5.13), we can evaluate the explicit structure of the left-invariant one form Λ. Using the $\mathcal{M}^{0|4\times\mathcal{N}}$ style of the Maurer Cartan equations (5.10) we obtain:

$$\Lambda \equiv \mathbb{L}^{-1}\,d\mathbb{L} = \left(\begin{array}{c|c} \Delta & \exp\left[-\mathrm{i}\frac{\pi}{4}\right]\,\Phi \\ \hline -4e\,\exp\left[-\mathrm{i}\frac{\pi}{4}\right]\,\Phi^T\,\epsilon & -\,e\,\mathcal{A} \end{array}\right) \tag{5.34}$$

where the 1-forms Δ, $\mathcal{A}$ and Φ can be explicitly calculated, using the explicit form of the inverse coset representative:

$$\mathbb{L}^{-1} = \left(\begin{array}{c|c} -\epsilon\,\mathcal{S}^T\,\epsilon & \exp\left[-\mathrm{i}\frac{\pi}{4}\right]\,\frac{1}{4e}\,\epsilon\Pi^T \\ \hline -\exp\left[-\mathrm{i}\frac{\pi}{4}\right]\,4e\,\Theta^T\,\epsilon & \mathcal{O}^T \end{array}\right) \tag{5.35}$$

$$\begin{aligned} e\mathcal{A} &= -\,\mathcal{O}^T\,d\mathcal{O} \,-\, \mathrm{i}\,4e\,\Theta^T\,\epsilon\,d\Theta \\ \Omega &= -\,\epsilon\,\mathcal{S}^T\,\epsilon\,d\mathcal{S} \,-\, \mathrm{i}\,\frac{1}{4e}\,\Pi^T\,d\Pi \\ \Phi &= -\,\epsilon\,S^T\,\epsilon\,d\Theta \,+\, \frac{1}{4e}\,\epsilon\,\Pi^T\,d\mathcal{O} \end{aligned} \tag{5.36}$$

5.4 The coset representative of $\mathrm{OSp}(\mathcal{N}|4)/\mathrm{Sp}(4)\times\mathrm{SO}(\mathcal{N})$

It is fairly simple to write an explicit form for the coset representative of the fermionic supermanifold

$$\mathcal{M}^{0|4\times\mathcal{N}} = \frac{\mathrm{OSp}(\mathcal{N}|4)}{\mathrm{Sp}(4,\mathbb{R})\times\mathrm{SO}(\mathcal{N})} \tag{5.37}$$

by adopting the upper left block components Θ of the supermatrix (5.32) as coordinates. It suffices to solve eqs. (5.33) for the sub blocks $\mathcal{S},\mathcal{O},\Pi$. Such an explicit solution is provided by setting:

$$\begin{aligned} \mathcal{O}(\Theta) &= \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta^T\,\epsilon\,\Theta\right)^{1/2} \\ \mathcal{S}(\Theta) &= \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta\,\Theta^T\,\epsilon\right)^{1/2} \\ \Pi &= 4e\left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta^T\,\epsilon\,\Theta\right)^{-1/2}\,\Theta^T\,\epsilon\,\left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta\,\Theta^T\,\epsilon\right)^{1/2} \\ &= 4e\,\Theta^T\,\epsilon \end{aligned} \tag{5.38}$$

In this way we conclude that the coset representative of the fermionic supermanifold (5.37) can be chosen to be the following supermatrix:

$$\mathbb{L}\,(\Theta) = \left(\begin{array}{c|c} \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta\,\Theta^T\,\epsilon\right)^{1/2} & \exp\left[-\,\mathrm{i}\frac{\pi}{4}\right]\Theta \\ \hline -\,\exp\left[-\,\mathrm{i}\frac{\pi}{4}\right]\,4e\,\Theta^T\,\epsilon & \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta^T\,\epsilon\,\Theta\right)^{1/2} \end{array}\right) \tag{5.39}$$

By straightforward steps from eq. (5.35) we obtain the inverse of the supercoset element (5.39) in the form:

$$\mathbb{L}^{-1}\,(\Theta) = \mathbb{L}\,(-\,\Theta) = \left(\begin{array}{c|c} \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta\,\Theta^T\,\epsilon\right)^{1/2} & -\,\exp\left[-\,\mathrm{i}\frac{\pi}{4}\right]\Theta \\ \hline \exp\left[-\,\mathrm{i}\frac{\pi}{4}\right]\,4e\,\Theta^T\,\epsilon & \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta^T\,\epsilon\,\Theta\right)^{1/2} \end{array}\right) \tag{5.40}$$

Correspondingly we work out the explicit expression of the Maurer Cartan forms:

$$\begin{aligned} e\mathcal{A} &= \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta^T\,\epsilon\,\Theta\right)^{1/2}\,d\left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta^T\,\epsilon\,\Theta\right)^{1/2} \,-\, \mathrm{i}4e\,\Theta^T\,\epsilon\,d\Theta \\ \Phi &= \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta\,\Theta^T\,\epsilon\right)^{1/2}\,d\Theta \,+\, \Theta\,d\left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta^T\,\epsilon\,\Theta\right)^{1/2} \\ \Delta &= \left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta\,\Theta^T\epsilon\right)^{1/2}\,d\,\left(\mathbf{1} \,+\, 4\,\mathrm{i}\,e\,\Theta\,\Theta^T\epsilon\right)^{1/2} \,-\, \mathrm{i}\,4e\,\Theta\,d\Theta^T\,\epsilon \end{aligned} \tag{5.41}$$

5.5 Gauged Maurer Cartan 1-forms of OSp(8|4)

A fundamental ingredient in the construction of gauged supergravities is constituted by the gauging of Maurer Cartan forms of the scalar coset manifold G/H (see for instance [47] for a survey of the subject). The vector fields present in the supermultiplet, which are 1-forms defined over the space-time manifold $\mathcal{M}_4$, are used to deform the Maurer Cartan 1-forms of the scalar manifold G/H that are instead sections of $T^\star\,(\mathrm{G/H})$. *Mutatis mutandis*, a similar construction turns out to be quite essential in the problem of gauge completion under consideration. In our case what will be gauged are the Maurer Cartan 1-forms of the supercoset (5.13) which contains the fermionic coordinates of the final superspace we desire to construct. The role of the space-time gauge fields is instead played by the **U**-connection (4.5) of the $\mathfrak{so}(8)$ spinor bundle constructed over the internal 7-manifold $(\mathrm{G/H})_7$.

Accordingly we define:

$$\widehat{\Lambda} \equiv \mathbb{L}^{-1}\,\nabla\,\mathbb{L} = \mathbb{L}^{-1}\,\left(d\,\mathbb{L} + \left[\widehat{\mathbf{U}}\,,\,\mathbb{L}\right]\right) \tag{5.42}$$

where $\widehat{\mathbf{U}}$ is the supermatrix defined by the canonical immersion of the $\mathfrak{so}(8)$ Lie algebra into the orthosymplectic superalgebra:

$$\begin{aligned}\widehat{\mathbf{U}} &= \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & \mathbf{U}\end{array}\right) = \mathcal{I}\,(\mathbf{U}) \\ \mathcal{I} &: \mathfrak{so}(8) \mapsto \mathfrak{osp}(8|4)\end{aligned} \tag{5.43}$$

As a result of their definition, the gauged Maurer Cartan forms satisfy the following deformed Maurer Cartan equations:

$$\nabla\widehat{\Lambda} + \widehat{\Lambda}\wedge\widehat{\Lambda} = \mathbb{L}^{-1}\,(\Theta)\,\left[\widehat{F[\mathbf{U}]}\,,\,\mathbb{L}\,(\Theta)\right] \tag{5.44}$$

where

$$\widehat{F[\mathbf{U}]} = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & F[\mathbf{U}]\end{array}\right) \tag{5.45}$$

By explicit evaluation, from eq. (5.44) we obtain the following deformation of the Maurer Cartan equations (5.10):

$$\begin{aligned} d\widehat{\Delta}^{xy} + \widehat{\Delta}^{xz}\wedge\widehat{\Delta}^{ty}\,\epsilon_{zt} + 4\,\mathrm{i}\,e\,\widehat{\Phi}^x_A\wedge\widehat{\Phi}^y_A, &= -\,\mathrm{i}\,\Theta^x_A\,F_{AB}[\mathbf{U}]\,\Theta^y_B \\ \nabla\widehat{\mathcal{A}}_{AB} - e\widehat{\mathcal{A}}_{AC}\wedge\widehat{\mathcal{A}}_{CB} - 4\,\mathrm{i}\widehat{\Phi}^x_A\wedge\widehat{\Phi}^y_B\,\epsilon_{xy} &= \mathcal{O}_{AP}(\Theta)\,F_{PQ}[\mathbf{U}]\,\mathcal{O}_{QB}(\Theta) - F_{AB}[\mathbf{U}] \\ d\widehat{\Phi}^x_A + \widehat{\Delta}^{xy}\wedge\epsilon_{yz}\,\widehat{\Phi}^z_A - e\,\widehat{\mathcal{A}}_{AB}\wedge\widehat{\Phi}^x_B &= \Theta^x_P\,F_{PQ}[\mathbf{U}]\,\mathcal{O}_{QA}(\Theta)\end{aligned} \tag{5.46}$$

The above equations will be our main starting point in the discussion of the supergauge completion for compactifications with less preserved supersymmetry.

5.6 Constrained superspace and the supersolvable parametrization

In [11] it was demonstrated that, in full analogy with the solvable parametrization of non compact bosonic coset manifolds, extensively utilized while dealing with the scalar sector of supergravity models, one can introduce also a supersolvable parametrization of the supermanifold $\mathcal{M}^{4|4\times\mathcal{N}}_{osp}$ defined in eq. (5.9) (see [48, 11]). This latter is the supergroup manifold of a solvable super Lie subalgebra $SSolv_{4|\mathcal{N}} \subset \mathrm{Osp}(\mathcal{N}|4)$. Similarly to the bosonic case the solvable parametrization of the supermanifold leads to an enormous simplification of the Maurer Cartan forms since the coset representative becomes polynomial in its parameters, yet differently from the bosonic case the supersolvable algebra $SSolv_{4|\mathcal{N}}$ has smaller dimension than the dimension of the original coset $\mathcal{M}^{4|4\times\mathcal{N}}_{osp}$. In other words the supergroup manifold:

$$\mathcal{SM}^{4|2\times\mathcal{N}} \equiv \exp\left[\,SSolv_{4|\mathcal{N}}\,\right] \tag{5.47}$$

does not contain all the Θ-coordinates but only a subset. Actually as it is implied by the chosen notation, the solvable supergroup manifold $\mathcal{SM}^{4|2\times\mathcal{N}}$ contains just one-half of the thetas, namely $2\times\mathcal{N}$. In [11] this was interpreted in terms of κ-supersymmetry. Indeed it was advocated that starting from the general κ-supersymmetric action of the $M2$-brane, one can localize it on an $\mathrm{AdS}_4 \times \mathbb{S}^7$ background in a form where all κ-supersymmetry are already gauged-fixed. This is the form taken by the general action when the Maurer Cartan forms of $\mathrm{Osp}(\mathcal{N}|4)$ are written in the supersolvable parametrization. Alternatively one realizes that the solvable super Lie algebra $SSolv_{4|\mathcal{N}}$ is nothing else but the $\mathcal{N}$-extended Poincaré superalgebra in three-space time dimensions, i.e. on the membrane world-volume, while the complete $\mathrm{Osp}(\mathcal{N}|4)$ algebra is simply the superconformal extension of such an algebra. Hence the supermanifold (5.47) is just the ordinary Poincaré superspace for field theories on the membrane and the used thetas are the superPoincaré ones while those deleted are the parameters of conformal supersymmetry which can be non linear realized on the Poincaré ones.

Explicitly the supersolvable parametrization works as follows. We look for a decomposition of the $\mathrm{Osp}(\mathcal{N}|4)$ algebra of the following form:

$$\mathrm{Osp}(\mathcal{N}|4) \;=\; (\mathrm{SO}(1,3)\otimes\mathrm{SO}(\mathcal{N})\otimes\mathcal{Q})\oplus SSolv_{4|\mathcal{N}}, \tag{5.48}$$

where $\mathcal{Q} = \left\{Q^A_-\right\}$ is a subset of the fermionic generators defined by a suitable projection operator $\mathcal{P}_\pm$

$$\begin{aligned} Q^A_- &= \mathcal{P}_-\cdot Q^A \\ Q^A_+ &= \mathcal{P}_+\cdot Q^A, \\ \mathcal{P}^2_\pm &= \mathcal{P}_\pm\ ;\ \mathcal{P}_+\cdot\mathcal{P}_- = 0. \end{aligned} \tag{5.49}$$

The main idea underlying the construction rules of the supersolvable algebra generating $\mathcal{SM}^{4|2\times\mathcal{N}}$ as well as the solvable algebra generating anti de Sitter space is that of *grading*. The Cartan generator contained in the coset of AdS_4 defines a partition of the isometry generators into eigenspaces corresponding to positive, negative or null eigenvalues

$(\mathbf{g}_{(\pm 1)}, \mathbf{sg}_{(\pm 1/2)}, \mathbf{sg}_{(0)})$ and the structure of the solvable and supersolvable algebras (*Solv* and *SSolv*) is the following:

$$\begin{aligned}
\mathbf{g} = \mathrm{SO}(2,3) \sim \mathrm{Sp}(4,\mathbb{R}) &\to \mathbf{g}_{(-1)} \oplus \mathbf{g}_{(0)} \oplus \mathbf{g}_{(+1)}, \\
Solv_4 &= \{\mathcal{C}\} \oplus \mathbf{g}_{(-1)}, \\
\mathbf{sg} = \mathrm{Osp}(\mathcal{N}|4) &\to \mathbf{g}_{(-1)} \oplus \mathbf{sg}_{(0)} \oplus \mathbf{g}_{(+1)} \oplus \mathbf{sg}_{(-1/2)} \oplus \mathbf{sg}_{(1/2)}, \\
\mathbf{sg}_{(0)} &= \mathbf{g}_{(0)} \oplus \mathrm{SO}(\mathcal{N}), \\
SSolv_{4|\mathcal{N}} &= \{\mathcal{C}\} \oplus \mathbf{g}_{(-1)} \oplus \mathbf{sg}_{(-1/2)},
\end{aligned} \tag{5.50}$$

where $\mathbf{sg}_{(\pm 1/2)}$ represents the grading induced by the Cartan generator on the fermionic isometries and the eigenspace $\mathbf{sg}_{(+1/2)}$ not entering the construction of *SSolv* is the space $\mathcal{Q} = \{Q^A_+\}$ in eq. (5.48) and generates the special conformal transformations. Moreover these generators on the chosen solution of the world volume theory, generate the local κ-supersymmetry transformations. As shown in [11] the projection operator which singles out the subspaces $\mathbf{sg}_{(\pm 1/2)}$ is simply given in terms of $4D$-gamma matrices as follows:

$$\begin{aligned}
\mathcal{P}_\pm &= \frac{1}{2}(\mathbb{1} \pm \gamma^5\gamma^2), \\
\mathbf{sg}_{(\pm 1/2)} &= \{Q^A_\pm\} = \{\mathcal{P}_\pm Q^A\}.
\end{aligned} \tag{5.51}$$

It is straightforward to verify that such a projection is compatible with the Majorana condition and it is immediate to solve such a constraint in the basis of gamma matrices described in appendix B.2. Indeed we find:

$$Q^A_\pm = \begin{pmatrix} Q^A_1 \\ Q^A_2 \\ \mp Q^A_2 \\ \pm Q^A_1 \end{pmatrix} \tag{5.52}$$

This implies that the corresponding Θ-coordinates have the same structure:

$$\Theta^A = \Theta^A_+ \oplus \Theta^A_- \quad ; \quad \Theta^A_\pm = \begin{pmatrix} \Theta^{A\pm}_1 \\ \Theta^{A\pm}_2 \\ \mp\Theta^{A\pm}_2 \\ \pm\Theta^{A\pm}_1 \end{pmatrix} \tag{5.53}$$

Next it can be immediately verified that the projected $\Theta.s$ satisfy the following constraints:

$$\Theta^x_A = \Theta^x_{A\pm} \tag{5.54}$$

$$\Downarrow$$

$$\Theta^x_A\, \Theta^y_B\, \epsilon_{xy} = 0 \quad \text{and} \quad \Theta^x_A\, d\Theta^y_B\, \epsilon_{xy} = \Theta^x_B\, d\Theta^y_A\, \epsilon_{xy} \tag{5.55}$$

As explained in the introduction, in this paper we take a different point of view. Rather then using the solvable parametrization we take the complete parametrization of the supercosets (either $\mathcal{M}^{4|4\times N}$ or $\mathcal{M}^{4|4\times N}$) but we enforce the constraints (5.55) on the fermionic coordinates cutting out a sixteen dimensional locus in the 32-dimensional one. In this way we preserve all the symmetries and yet we obtain a formidable simplification of the Maurer Cartan forms which allows to pursue the gauge completion programme to its very end.

5.7 Gauged Maurer Cartan forms in constrained superspace

Let us now consider the consequences of the constraints (5.55) on the coset representative (5.39), the Maurer Cartan forms (5.41) and their gauged counterparts (5.42). On the constrained surface we immediately find:

$$\begin{aligned}
\mathcal{O}(\Theta) &= 1 \\
\mathcal{S}(\Theta) &= \mathbf{1} + 2\,\mathrm{i}\,e\,\Theta\,\Theta^T\,\epsilon \\
\widehat{\mathcal{A}} &= \mathcal{A} = 0 \\
\widehat{\Delta} &= 2\,\mathrm{i}\,\left(\nabla\Theta\,\Theta^T - \Theta\,\nabla\Theta^T\right)
\end{aligned} \tag{5.56}$$

and the gauged Maurer Cartan equations (5.46) become:

$$\begin{aligned}
d\widehat{\Delta}^{xy} + \widehat{\Delta}^{xz} \wedge \widehat{\Delta}^{ty}\,\epsilon_{zt} + 4\,\mathrm{i}\,e\,\widehat{\Phi}^x_A \wedge \widehat{\Phi}^y_A, &= -\,\mathrm{i}\,\Theta^x_A\,F_{AB}[\mathbf{U}]\,\Theta^y_B \\
0 &= 0 \\
d\widehat{\Phi}^x_A + \widehat{\Delta}^{xy} \wedge \epsilon_{yz}\,\widehat{\Phi}^z_A &= \Theta^x_P\,F_{PA}[\mathbf{U}]
\end{aligned} \tag{5.57}$$

As we are going to show in the sequel, the above equations enable us to write a complete parametrization of all the FDA superforms adapted to any background $\mathrm{AdS}_4 \times (\mathcal{G}/\mathcal{H})_7$.

6. Killing spinors of the AdS_4 manifold

The next main item for the construction of the supergauge completion is given by the Killing spinors of anti de Sitter space. Indeed, in analogy with the Killing spinors of the internal 7-manifold, defined by eq. (3.30) with $m = 1$, we can now introduce the notion of Killing spinors of the AdS_4 space and recognize how they can be constructed in terms of the coset representative, namely in terms of the fundamental harmonic of the coset.

The analogue of eq. (3.30) is given by:

$$\nabla^{\mathrm{Sp}(4)}\,\chi_x \equiv \left(d - \frac{1}{4}\,B^{ab}\,\gamma_{ab} - 2\,e\,\gamma_a\,\gamma_5\,B^a\right)\chi_x = 0 \tag{6.1}$$

and states that the Killing spinor is a covariantly constant section of the $\mathfrak{sp}(4,\mathbb{R})$ bundle defined over AdS_4. This bundle is flat since the vanishing of the $\mathfrak{sp}(4,\mathbb{R})$ curvature is nothing else but the Maurer Cartan equation of $\mathfrak{sp}(4,\mathbb{R})$ and hence corresponds to the structural equations of the AdS_4 manifold. We are therefore guaranteed that there exists a basis of four linearly independent sections of such a bundle, namely four linearly independent solutions of eq. (6.1) which we can normalize as follows:

$$\overline{\chi}_x\,\gamma_5\,\chi_y = \mathrm{i}\,(\mathcal{C}\,\gamma_5)_{xy} \tag{6.2}$$

Let $\mathrm{L_B}$ the coset representative mentioned in eq. (5.18) and satisfying:

$$-\frac{1}{4}\,B^{ab}\,\gamma_{ab} - 2\,e\,\gamma_a\,\gamma_5\,B^a = \Delta_B = \mathrm{L_B^{-1}}\,d\mathrm{L_B} \tag{6.3}$$

It follows that the inverse matrix $\mathrm{L}_{\mathrm{B}}^{-1}$ satisfies the equation:

$$(d + \Delta_B)\, \mathrm{L}_{\mathrm{B}}^{-1} = 0 \tag{6.4}$$

Regarding the first index y of the matrix $\left(\mathrm{L}_{\mathrm{B}}^{-1}\right)^{y}{}_{x}$ as the spinor index acted on by the connection Δ_B and the second index x as the labeling enumerating the Killing spinors, eq. (6.4) is identical with eq. (6.1) and hence we have explicitly constructed its four independent solutions. In order to achieve the desired normalization (6.2) it suffices to multiply by a phase factor $\exp\left[-\mathrm{i}\,\frac{1}{4}\pi\right]$, namely it suffices to set:

$$\chi^{y}_{(x)} = \exp\left[-\mathrm{i}\,\frac{1}{4}\pi\right]\, \left(\mathrm{L}_{\mathrm{B}}^{-1}\right)^{y}{}_{x} \tag{6.5}$$

In this way the four Killing spinors fulfill the Majorana condition. Furthermore since $\mathrm{L}_{\mathrm{B}}^{-1}$ is symplectic it satisfies the defining relation

$$\mathrm{L}_{\mathrm{B}}^{-1}\, \mathcal{C}\, \gamma_5\, \mathrm{L}_{\mathrm{B}} = \mathcal{C}\, \gamma_5 \tag{6.6}$$

which implies (6.2).

7. Supergauge completion in mini superspace

As it was observed many years ago in [29, 44] and it is reviewed at length in the book [23], given a bosonic Freund Rubin compactification of D=11 supergravity on an internal coset manifold $\mathcal{M}_7 = \frac{\mathcal{G}}{\mathcal{H}}$ which admits $\mathcal{N}$ Killing spinors it is fairly easy to extend it consistently to a *mini-superspace* $\mathcal{M}^{11|4\times\mathcal{N}}$ which contains all of the eleven bosonic coordinates but only $4 \times \mathcal{N}$ θ.s, namely those which are associated with unbroken supersymmetries. We review this extension reformulating it in such a way that it is suitable for its generalization to all θ.s namely also to those associated with broken supersymmetries.

In the original formulation, the mini superspace is viewed as the following tensor product

$$\mathcal{M}^{11|4\times\mathcal{N}} \equiv \mathcal{M}^{4|4\mathcal{N}}_{osp} \times \frac{\mathcal{G}}{\mathcal{H}} \tag{7.1}$$

and in order to construct the FDA p-forms, in addition to the Maurer Cartan forms of the above coset, we just need to introduce the Killing spinors of the bosonic internal manifold. Let η^A be an orthonormal basis of $\mathcal{N}$ eight component Killing spinors satisfying the Killing spinor condition (3.31) and the normalization:

$$\left(\eta^{\underline{A}}\right)^T \eta^{\underline{B}} = \delta^{\underline{AB}} \tag{7.2}$$

Next, following [23] and [11], whose results were also summarized in [4], we can now write the complete solution for the background fields in the case of $\mathrm{AdS}_4 \times \frac{\mathcal{G}}{\mathcal{H}}$ Freund-Rubin backgrounds:

$$
\begin{aligned}
\widehat{V}^{\underline{a}} &= \begin{cases} \widehat{V}^{a} = E^{a} \\ \widehat{V}^{\alpha} = \mathcal{B}^{\alpha} - \frac{1}{8}\,\overline{\eta}_{\underline{A}}\,\tau^{\alpha}\,\eta_{\underline{B}}\,\mathcal{A}_{AB} \end{cases} \\
\widehat{\omega}^{\underline{ab}} &= \begin{cases} \widehat{\omega}^{ab} = \omega^{ab} \\ \widehat{\omega}^{\alpha b} = 0 \\ \widehat{\omega}^{\alpha\beta} = \mathcal{B}^{\alpha\beta} + \frac{e}{4}\,\overline{\eta}_{\underline{A}}\,\tau^{\alpha\beta}\,\eta_{\underline{B}}\,\mathcal{A}_{\underline{AB}} \end{cases} \\
\widehat{\Psi} &= \eta_{\underline{A}} \otimes \psi_{\underline{A}}
\end{aligned}
\tag{7.3}
$$

where $\left\{\mathcal{B}^{\alpha\beta}\,,\,\mathcal{B}^{\alpha}\right\}$ are the spin connection and the vielbein, respectively, of the bosonic seven dimensional coset manifold $\frac{\mathcal{G}}{\mathcal{H}}$.

Let us now observe that in this formulation of the superextension, the fermionic coordinates are actually attached to the space-time manifold AdS_4, which is superextended to a supercoset manifold:

$$
\mathrm{AdS}_4 \overset{\text{superextension}}{\Longrightarrow} \frac{\mathrm{Osp}(\mathcal{N}\,|\,4)}{\mathrm{SO}(\mathcal{N}) \times \mathrm{SO}(1,3)} \equiv \mathcal{M}^{4|4\times\mathcal{N}}
\tag{7.4}
$$

At the same time the internal manifold $\mathcal{M}_7 = \frac{\mathcal{G}}{\mathcal{H}}$ is regarded as purely bosonic and it is twisted into the fabric of the Free Differential Algebra through the notion of the Killing spinors η_A, defined as covariantly constant sections of the SO(8) spinor bundle over $\mathcal{M}_7$.

Yet whether supersymmetries are preserved or broken precisely depends on the structure of the SO(8) spinor bundle on $\mathcal{M}_7$. Henceforth it is suggestive to think that the fermionic coordinates should not be attached to either the internal or to external manifold, rather they should live as a fiber over the bosonic manifolds. The first step in order to realize such a programme consists of a reformulation of the superextension in minisuperspace that treats the space-time manifold AdS_4 and the internal manifold $\mathcal{M}_7$ in a symmetric way and in both instances relies on the notion of Killing spinors of the bosonic submanifold as a way of including the fermionic one. This can be easily done in view of eq. (5.14) whose precise meaning we have explained in section 5.2. Indeed in view of eq. (5.14) we can look at at eq. (7.1) in the following equivalent, but more challenging fashion:

$$
\begin{aligned}
\mathcal{M}^{11|4\times\mathcal{N}} &= \mathrm{AdS}_4 \times \mathcal{M}^{0|4\times\mathcal{N}} \times \mathcal{M}_7 \\
&\equiv \underbrace{\frac{\mathrm{Sp}(4,\mathbb{R})}{\mathrm{SO}(1,3)}}_{\mathrm{AdS}_4} \times \underbrace{\frac{\mathrm{Osp}(\mathcal{N}\,|\,4)}{\mathrm{SO}(\mathcal{N}) \times \mathrm{Sp}(4,\mathbb{R})}}_{4\times\mathcal{N}\,\text{fermionic manifold}} \times \underbrace{\frac{\mathcal{G}}{\mathcal{H}}}_{\mathcal{M}_7}
\end{aligned}
\tag{7.5}
$$

The above equation simply corresponds to the rewriting of eq. (7.3) in the following way

$$
\begin{aligned}
\widehat{V}^{\underline{a}} &= \begin{cases} \widehat{V}^{a} = B^{a} - \frac{1}{8\,e}\,\overline{\chi}_x\,\gamma^{a}\,\chi_y\,\Delta_F^{xy} \\ \widehat{V}^{\alpha} = \mathcal{B}^{\alpha} - \frac{1}{8}\,\overline{\eta}_{\underline{A}}\,\tau^{\alpha}\,\eta_{\underline{B}}\,\mathcal{A}_{\underline{AB}} \end{cases} \\
\widehat{\omega}^{\underline{ab}} &= \begin{cases} \widehat{\omega}^{ab} = B^{ab} + \frac{1}{2}\,\overline{\chi}_x\,\gamma_5\,\gamma^{ab}\,\chi_y\,\Delta_F^{xy} \\ \widehat{\omega}^{\alpha b} = 0 \\ \widehat{\omega}^{\alpha\beta} = \mathcal{B}^{\alpha\beta} + \frac{e}{4}\,\overline{\eta}_{\underline{A}}\,\tau^{\alpha\beta}\,\eta_{\underline{B}}\,\mathcal{A}_{\underline{AB}} \end{cases} \\
\widehat{\Psi} &= \eta_{\underline{A}} \otimes \chi_x\,\Phi^{x|\underline{A}}
\end{aligned}
\tag{7.6}
$$

8. Gauge completion in the full constrained superspace

We are now in a position to write an ansatz which solves the rheonomic parametrization of the FDA curvatures for any $\mathrm{AdS}_4 \times (\mathcal{G}/\mathcal{H})_7$ back ground and involves all the Θ-coordinates although constrained. The extension to mini-superspace provided by eqs. (7.6) is our starting point. In those equations the Maurer Cartan forms are those (*ungauged*) of the supermanifold:

$$\frac{\mathrm{Osp}(\mathcal{N}|4)}{\mathrm{Sp}(4,\mathbb{R}) \times \mathrm{SO(N)}} \tag{8.1}$$

and therefore are written in terms of $4 \times \mathcal{N}$ *unconstrained* fermionic coordinates. The summation on the indices $\underline{A, B, C}$ is on $\mathcal{N}$-values since $\eta_{\underline{A}}$ are just the Killing spinors. The MC-forms are ungauged because, by definition, there is no **U**-connection in the directions spanned by the Killing spinors.

The new solution in complete constrained superspace has the following form:

$$\begin{aligned}
\widehat{V}^{\underline{a}} &= \begin{cases} \widehat{V}^a = B^a - \frac{1}{8\,e}\,\overline{\chi}_x\,\gamma^a\,\chi_y\,\widehat{\Delta}^{xy}_F \\ \widehat{V}^\alpha = \mathcal{B}^\alpha - \frac{1}{8}\,\overline{\zeta}_A\,\tau^\alpha\,\zeta_B\,\widehat{\mathcal{A}}_{AB} = \mathcal{B}^\alpha \end{cases}\\
\widehat{\omega}^{\underline{ab}} &= \begin{cases} \widehat{\omega}^{ab} = B^{ab} + \frac{1}{2}\,\overline{\chi}_x\,\gamma_5\,\gamma^{ab}\,\chi_y\,\widehat{\Delta}^{xy}_F \\ \widehat{\omega}^{\alpha b} = \Delta\omega^{\alpha b} \\ \widehat{\omega}^{\alpha\beta} = \mathcal{B}^{\alpha\beta} + \frac{e}{4}\,\overline{\zeta}_A\,\tau^{\alpha\beta}\,\zeta_B\,\widehat{\mathcal{A}}_{AB} = \mathcal{B}^{\alpha\beta} + \Delta\omega^{\alpha\beta} \end{cases}\\
\widehat{\Psi} &= \zeta_A \otimes \chi_x\,\widehat{\Phi}^{x|A}
\end{aligned} \tag{8.2}$$

The modifications that have occurred with respect to eq. (7.6) are the following ones:

1. The indices A, B, C run on 8-values and rather then the Killing spinors η_A we have a complete basis of sections ζ_A of the $\mathfrak{so}(8)$ spin bundle.

2. The MC forms are those of the supermanifold

$$\frac{\mathrm{Osp}(8|4)}{\mathrm{Sp}(4,\mathbb{R}) \times \mathrm{SO}(8)} \tag{8.3}$$

   but they are not the ordinary ones, $\mathcal{A}, \Delta, \Phi$, rather those gauged by means of the **U**-connection on the $\mathfrak{so}(8)$-spinor bundle over $\mathcal{G}/\mathcal{H}$. This is signaled by the hat: $\widehat{\mathcal{A}}, \widehat{\Delta}, \widehat{\Phi}$.

3. The 32 coordinates of the supermanifold (8.3) are not free, rather they are subject to the constraints (5.55). This implies in particular that $\widehat{\mathcal{A}}$ vanishes.

4. The spin connection contains a correction term which is due to the gauging and which we easily calculate below. In particular due to this correction the mixed components $\omega^{\alpha b}$ are no longer zero.

It is fairly easy to verify by direct evaluation that the ansatz (8.2) verifies the torsion equation (2.5) and the gravitino equation (2.8). The mixed part of the spin connection is just a consequence of the F-deformation of the Maurer Cartan equation appearing in the first of eqs. (5.57). By explicit evaluation we find that without introducing the correction $\Delta\omega^{\underline{ab}}$ the torsion is not zero, rather it is given by:

$$T^{\mathrm{a}} = \begin{cases} T^{a} = \mathrm{cost}\,\overline{\chi}_x\,\gamma^a\chi_y\,\zeta_A\,\tau_{\rho\sigma}\zeta_B\,\Theta^x_A\,\Theta^y_B\,\mathcal{C}^{\rho\sigma}{}_{\alpha\beta}\,\mathcal{B}^\alpha \wedge \mathcal{B}^\beta \\ T^\alpha = 0 \end{cases} \tag{8.4}$$

In view of the parametrization (8.2) this means that the torsion is of the form:

$$\begin{aligned} T^{\underline{a}} &= H^{\underline{a}|\underline{bc}}\,V_b \wedge V_{\underline{c}} \\ H^{a|\beta\gamma} &= \mathrm{cost}\,\overline{\chi}_x\,\gamma^a\chi_y\,\zeta_A\,\tau_{\rho\sigma}\zeta_B\,\Theta^x_A\,\Theta^y_B\,\mathcal{C}^{\rho\sigma}{}_{\alpha\beta} \\ \text{all other components of } H^{\underline{a}|\underline{bc}} &= 0 \end{aligned} \tag{8.5}$$

which can be reabsorbed by the following redefinition of the spin connection:

$$\begin{aligned} \omega^{\underline{ab}} &\mapsto \omega^{\underline{ab}} + \Delta\omega^{\underline{ab}} \\ \Delta\omega^{\underline{ab}} &= -\left(H^{\underline{a}|\underline{bc}} - H^{\underline{b}|\underline{ac}} - H^{\underline{c}|\underline{ab}}\right) V^{\underline{c}} \end{aligned} \tag{8.6}$$

8.1 The 3-form

We have found an explicit expression for the supervielbein $V^{\underline{a}}$, the gravitino 1-form Ψ and ant he spin-connection $\omega^{\underline{ab}}$. In order to complete the description of the superextension we need also to provide an expression for the 3-form $A^{[3]}$. According to the general definitions of the FDA curvatures eq. (2.3) and the rheonomic parametrization (2.6) we find that:

$$d\mathbf{A}^{[3]} = \mathbf{F}^{[4]} - \frac{1}{2}\overline{\Psi} \wedge \Gamma_{\underline{ab}}\,\Psi \wedge V^{\underline{a}} \wedge V^{\underline{b}} \tag{8.7}$$

$$\Downarrow$$

$$\begin{aligned} d\mathbf{A}^{[3]} = {} & e\,\epsilon_{abcd}\,E^a \wedge E^b \wedge E^c \wedge E^d + \frac{1}{2}\overline{\chi}_x\,\gamma_{ab}\,\chi_y\,\Phi^x_A \wedge \Phi^y \wedge E^a \wedge E^b \\ & + \frac{1}{2}\overline{\chi}_x\,\chi_y\,\zeta_A\,\tau_{\alpha\beta}\,\zeta_B\,\Phi^x_A \wedge \Phi^y_B \wedge \mathcal{B}^\alpha \wedge \mathcal{B}^\beta \\ & + \overline{\chi}_x\,\gamma_a\,\gamma_5\chi_y\,\zeta_A\,\tau_\beta\,\zeta_B\,\Phi^x_A \wedge \Phi^y_B \wedge E^a \wedge \mathcal{B}^\beta \end{aligned} \tag{8.8}$$

The expression of $dA^{[3]}$ as a 4-form is completely explicit in eq. (8.8) and by construction it is integrable in the sense that $d^2\mathbf{A}^{[3]} = 0$. One might desire to solve this equation by finding a suitable expression for $A^{[3]}$ such that eq. (8.8) is satisfied. This is not possible in general terms, namely by using only the invariant constraints (5.55). In order to find explicit solutions, one needs to use some explicit coordinate system and some explicit solution of the constraints. For instance using the solvable parametrization it was shown in paper ([11]) how to write $A^{[3]}$ in the case of the seven sphere. This analysis could be pursued also for the other instances of compactifications with less supersymmetry, but it is not in the spirit we have adopted. Here it is just the constraints what matters, not their explicit solutions. In the main application we have in mind, namely while localizing the pure spinor BRST invariant action of the supermembrane M2 on such backgrounds, we can easily avoid all such problems. We simply substitute the world volume integral of $A^{[3]}$ with:

$$\int_{WV_3} A^{[3]} \mapsto \int_{WV_4} dA^{[3]} \tag{8.9}$$

where the 4-dimensional integration volume WV_4 is such that its boundary is the original supermembrane world-volume:

$$tial\, WV_4 \;=\; WV_3 \tag{8.10}$$

and we circumvent the problem of solving eq. (8.8).

With this observation we have concluded our proof that any $\mathrm{AdS}_4 \times \mathcal{G}/\mathcal{H}$ bosonic solution of D=11 supergravity field equations can be explicit gauge completed to a solution in a constrained superspace containing all the theta variables both associated with unbroken as with with broken supersymmetries. Such a superspace extension is just suited for the pure spinor action of the M2 brane as derived in [4].

9. Conclusions

The problem addressed in this paper is the supergauge completion of D=11 supergravity backgrounds of the form $\mathrm{AdS}_4 \times (\mathcal{G}/\mathcal{H})$. In short this corresponds to deriving an explicit parametrization of the p-forms of D=11 supergravity FDA in terms of all 32 fermionic coordinates plus the 11 bosonic coordinates of the chosen manifold $\mathrm{AdS}_4 \times (\mathcal{G}/\mathcal{H})$. The main motivation of solving such a problem is that the searched parametrization provides the necessary information in order to convert the general pure spinor action of the M2 brane derived in [4] into an explicit form.

Our solution is based on three ingredients: 1) identification of the obstruction which breaks supersymmetry in the non-trivial curvature of an $SO(8)$ connection $\mathbf{U}$ over the spinor bundle of the internal manifold G/H; 2) the replacement of $Osp(8|4)$ Maurer-Cartan forms with their gauged counterparts by means of the U-connection; 3) the implementation of a quadratic constraint on the θ coordinates which in particular admits the solvable parametrization of supercoset manifold previously discussed in [11].

It is rather straightforward that the same ingredients can be used for superstrings in the less-supersymmetric backgounds of AdS-type. We leave this subject to a forthcoming publication. Nevertheless, in the pure spinor formulation, one needs to BRST transform the constraints (5.55) into constraints for the pure spinors. We notice that by solving (5.55) we select a set of independent θ's. Their BRST variation provides a set of unconstrained commuting spinors on which we can still impose the pure spinor constraints. In this way we maintain the balance of degrees of freedom needed to cancel the conformal central charge. As a last remark, we point out that the target space supersymmetry is realized in a non-linear way and therefore the theory will be manifestly supersymmetric. These consideration will be presented more extensively in forthcoming publications.

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A. Index conventions

Due to the complexity of the Lie algebra and super Lie algebra structures which have to be intertwined together into a single fabric in order to produce our solution of the FDA equations, we are forced to introduce a plethora of different notations for different set of indices and in the present appendix we summarize our index conventions for the reader's benefit.

We distinguish two sets of index conventions: those relative to the general theory applying to a generic compactification on $\mathrm{AdS}_4 \times \mathrm{M}_7$ and those relative to the specific example $\mathcal{M}_7 = \mathcal{N}'^{\infty\prime}$

A.1 Index conventions for the general theory

We recall that all our indices are flat since we systematically use differential forms. Furthermore we have tried to incorporate consistently into our framework the index conventions adopted in the series of papers ([29, 44, 23, 26, 32 – 34]), dating back to the eighties and relative to the classification and construction of Freund Rubin compactifications and readopted in the series of papers ([40 – 43, 49]) relative to the reinterpretation of such solutions into the context of the AdS/CFT correspondence.

1. The underlined lower latin indices from the beginning of the alphabet $\underline{a, b, c}, \ldots = 0, 1, \ldots, 10$ run on eleven values and span the vector representation of the $\mathfrak{so}(1, 10)$ Lie algebra, namely the tangent Lie algebra of $D = 11$ supergravity.

2. The lower latin indices from the beginning of the alphabet $a, b, c, \ldots = 0, 1, \ldots, 3$ (without underlining) run on four values and span the vector representation of the $\mathfrak{so}(1, 3)$ Lie algebra, namely the tangent Lie algebra to the $D = 4$ space-time, specifically AdS_4.

3. The lower case greek indices from the beginning of the alphabet $\alpha, \beta, \gamma, \ldots = 1, \ldots, 7$ run on seven values and span the vector representation of the $\mathfrak{so}(7)$ Lie algebra namely the tangent Lie algebra to the internal seven manifold $\mathcal{M}_7$.

4. The capital latin indices $A, B, C, \ldots = 1, \ldots, 8$ from the beginning of the alphabet run on eight values and span the vector representation of $\mathfrak{so}(8)$. They enumerate the members of an orthonormal basis of sections $\{\zeta_A\}$ of the spinor bundle on $\mathcal{M}_7$.

5. Slightly modifying the general conventions of papers [29, 29, 44, 26, 32, 34], the underlined capital latin indices from the beginning of the alphabet $\underline{A, B, C}, \ldots$ run on $\mathcal{N}$ values and are the vector indices of the subgroup $\mathrm{SO}(\mathcal{N}) \subset \mathrm{Osp}(\mathcal{N}|4)$. They enumerate the members of an orthonormal basis of Killing spinors $\eta_{\underline{A}}$.

6. Hence we have in general:

$$\underline{a} = \left\{ \underbrace{a}_{4 \text{ values}} \quad , \quad \underbrace{\alpha}_{7 \text{ values}} \right\} \qquad A = \left\{ \underbrace{\underline{A}}_{\mathcal{N} \text{ values}} \quad , \quad \underbrace{\overline{B}}_{8\text{ -}\mathcal{N} \text{ values}} \right\} \tag{A.1}$$

7. The lower case latin indices from the end of the alphabet $x, y, z, t, \ldots$ take four values and are symplectic indices in the fundamental representation of $\mathfrak{sp}(4, R)$. They enumerate the members χ_x of an orthonormal basis of Killing spinors on the manifold AdS_4.

B. Spinor identities

In this section we list some spinor identities which are very useful in deriving various results discussed in the main text.

B.1 D=7 gamma matrix basis and spinor identities

We begin by writing the explicit form of the τ matrices used in the Kaluza-Klein supergravity literature [29] and in particular in the literature concerning the N^{010} manifold.[2]

The Clifford algebra:

$$\{\tau_\alpha \, , \, \tau_\beta\} \; = \; -\delta_{\alpha\beta} \tag{B.1}$$

is satisfied by the following, real, antisymmetric matrices:

$$\tau_1 \; = \; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \; ; \; \tau_2 \; = \; \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tau_3 \; = \; \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \; ; \; \tau_4 \; = \; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\tau_5 \; = \; \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \end{pmatrix} \; ; \; \tau_6 \; = \; \begin{pmatrix} 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \end{pmatrix}$$

$$\tau_7 \; = \; \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{B.2}$$

[2] Note that there is a change of basis with respect to the tau matrices used in paper [4]

Let $\zeta_{\underline{A}}$ be an orthonormal basis of section for the spinor bundle on $\mathcal{M}_7$, namely:

$$\overline{\zeta}_{\underline{A}}\,\zeta_{\underline{B}} = \delta_{\underline{AB}} \tag{B.3}$$

Now let $Q_{\underline{AB}} = -Q_{\underline{BA}}$ be any SO(8) Lie algebra valued 1-form and let us define the following objects:

$$\begin{aligned}
\Delta^{\alpha\beta} &\equiv \zeta_{\underline{A}}\tau^{\alpha\beta}\,\zeta_{\underline{B}}\,Q^{AB}\\
\Theta^{\alpha} &\equiv \zeta_{\underline{A}}\tau^{\alpha}\,\zeta_{\underline{B}}\,Q^{AB}\\
\Xi^{\alpha} &\equiv \zeta_{\underline{A}}\tau^{\alpha}\,\zeta_{\underline{B}}\,Q^{AC}\wedge Q^{CD}\\
\Pi^{\alpha\beta} &\equiv \zeta_{\underline{A}}\tau^{\alpha\beta}\,\zeta_{\underline{B}}\,Q^{AC}\wedge Q^{CD}
\end{aligned} \tag{B.4}$$

Then using the negative metric to saturate the $\mathfrak{so}(7)$ vector indices, as it is appropriate in our conventions, we find the following identities:

$$\begin{aligned}
\left(-\frac{1}{16}\,\Delta^{\alpha\beta}\,\tau_{\alpha\beta} + \frac{1}{8}\,\Theta^{\alpha}\,\tau_{\alpha}\right)\,\zeta_{\underline{A}} &= Q_{\underline{AB}}\,\zeta_{\underline{B}}\\
\Delta^{\alpha\beta}\wedge\Theta^{\beta} &= 4\,\Xi^{\alpha}\\
-\,\Delta^{\alpha\beta}\wedge\Delta^{\beta\gamma} &= -4\,\Pi^{\alpha\beta} + \Theta^{\alpha}\wedge\Theta^{\beta}
\end{aligned} \tag{B.5}$$

Next we consider the spinor identities in 4-dimensions.

B.2 D=4 γ-matrix basis and spinor identities

In this section we construct a basis of $\mathfrak{so}(1,3)$ gamma matrices such that it explicitly realizes the isomorphism $\mathfrak{so}(2,3) \sim \mathfrak{sp}(4,\mathbb{R})$ with the conventions used in the main text. Naming σ_i the standard Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i}\\ \mathrm{i} & 0\end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix} \tag{B.6}$$

we realize the $\mathfrak{so}(1,3)$ Clifford algebra:

$$\{\gamma_a\,,\,\gamma_b\} = 2\,\eta_{ab} \quad ; \quad \eta_{ab} = \mathrm{diag}\,(+,-,-,-) \tag{B.7}$$

by setting:

$$\begin{aligned}
\gamma_0 &= \sigma_2\otimes\mathbf{1}\ ;\ \gamma_1 = \mathrm{i}\,\sigma_3\otimes\sigma_1\\
\gamma_2 &= \mathrm{i}\sigma_1\otimes\mathbf{1}\ ;\ \gamma_3 = \mathrm{i}\sigma_3\otimes\sigma_3\\
\gamma_5 &= \sigma_3\otimes\sigma_2\ ;\ \mathcal{C} = \mathrm{i}\sigma_2\otimes\mathbf{1}
\end{aligned} \tag{B.8}$$

where γ_5 is the chirality matrix and $\mathcal{C}$ is the charge conjugation matrix. Making now reference to eqs. (5.2) and (5.3) of the main text we see that the antisymmetric matrix entering the definition of the orthosymplectic algebra, namely $\mathcal{C}\,\gamma_5$ is the following one:

$$\mathcal{C}\,\gamma_5 = \mathrm{i}\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\end{pmatrix} \tag{B.9}$$

namely it is proportional, through an overall i-factor, to a real completely off-diagonal matrix. On the other hand all the generators of the $\mathfrak{so}(2,3)$ Lie algebra, i.e. γ_{ab} and $\gamma_a\,\gamma_5$ are real, symplectic 4×4 matrices. Indeed we have

$$
\begin{aligned}
\gamma_{01} &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \; ; \; \gamma_{02} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
\gamma_{12} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \; ; \; \gamma_{13} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\
\gamma_{23} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \; ; \; \gamma_{34} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\
\gamma_0\,\gamma_5 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \; ; \; \gamma_1\,\gamma_5 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
\gamma_2\,\gamma_5 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \; ; \; \gamma_3\,\gamma_5 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}
\end{aligned}
\tag{B.10}
$$

On the other hand we find that $\mathcal{C}\gamma_0 = \mathrm{i}\,\mathbf{1}$. Hence the Majorana condition becomes:

$$\mathrm{i}\,\psi \;=\; \psi^\star \tag{B.11}$$

so that a Majorana spinor is just a real spinor multiplied by an overall phase $\exp\left[-i\frac{\pi}{4}\right]$.

These conventions being fixed let χ_x $(x = 1,\ldots,4)$ be a set of (commuting) Majorana spinors normalized in the following way:

$$
\begin{array}{lll}
\chi_x & = \mathcal{C}\,\overline{\chi}_x^T & ; \text{Majorana condition} \\
\overline{\chi}_x\,\gamma_5\,\chi_y & = \mathrm{i}\;(\mathcal{C}\,\gamma_5)_{xy} & ; \text{symplectic normal basis}
\end{array}
\tag{B.12}
$$

Then by explicit evaluation we can verify the following Fierz identity:

$$\frac{1}{2}\,\gamma^{ab}\,\chi_z\,\overline{\chi}_x\,\gamma_5\,\gamma_{ab}\,\chi_y \;-\; \gamma_a\,\gamma_5\,\chi_z\,\overline{\chi}_x\,\gamma_a\,\chi_y \;=\; -\,2\mathrm{i}\,\left[(C\gamma_5)_{zx}\,\chi_y \;+\; (C\gamma_5)_{zy}\,\chi_x\right] \tag{B.13}$$

Another identity which we can prove by direct evaluation is the following one:

$$
\begin{aligned}
&\overline{\chi}_x\,\gamma_5\gamma_{ab}\,\chi_y\,\overline{\chi}_z\,\gamma^b\,\chi_t \;-\; \overline{\chi}_z\,\gamma_5\gamma_{ab}\,\chi_t\,\overline{\chi}_x\,\gamma^b\,\chi_y = \\
&\quad \mathrm{i}\left(\overline{\chi}_x\,\gamma_a\,\chi_t\;(\mathcal{C}\,\gamma_5)_{yz} \;+\; \overline{\chi}_y\,\gamma_a\,\chi_t\;(\mathcal{C}\,\gamma_5)_{xz} + \overline{\chi}_x\,\gamma_a\,\chi_z\;(\mathcal{C}\,\gamma_5)_{yt} \;+\; \overline{\chi}_y\,\gamma_a\,\chi_z\;(\mathcal{C}\,\gamma_5)_{xt}\right)
\end{aligned}
\tag{B.14}
$$

Both these identities are of high relevance in our discussion of the supergauge completion.

C. The explicit form of the U-connection in a pair of examples

Since the central item in deriving the gauge superextension is provided by the **U**-connection on the $\mathfrak{so}(8)$ spinor bundle, it is appropriate to spell out the explicit form of a such a 1-form at least in a couple of cases. To this effect we shall consider the spaces Q^{111} and N^{010}.

C.1 The Q^{111} sasakian manifold

The 7 manifold Q^{111} is an $\mathbb{S}^1$ fibration over the product of three $\mathbb{P}^1$:

$$Q^{111} \stackrel{\pi}{\Longrightarrow} \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \tag{C.1}$$

the fibration being:

$$Q^{111} \sim \mathcal{O}\left(\mathbb{P}^1, 1\right) \otimes \mathcal{O}\left(\mathbb{P}^1, 1\right) \otimes \mathcal{O}\left(\mathbb{P}^1, 1\right) \tag{C.2}$$

This means that, as a coset manifold, it can be described as the particular instance $(p, q, r) = (1, 1, 1)$ in the infinite family of homogeneous spaces:

$$Q^{pqr} = \frac{\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_3 \times \mathrm{U}(1)}{\mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)} \tag{C.3}$$

$$Z = p\, J^3_{(1)} + q\, J^3_{(2)} + r\, J^3_{(2)} + Y \tag{C.4}$$

by definition Z being the Cartan generator that is not in the subalgebra $\mathrm{H} = \mathrm{U}(1) \times \mathrm{U}(1) \times \mathrm{U}(1)$, $J^a_{(i)}$ $(a = 1, 2, 3)$ being the generators of $\mathrm{SU}(2)_i$ and the hypercharge Y being the generator of $\mathrm{U}(1)$ in the numerator group G.

These 7-manifolds were originally introduced in [50] and their role as solutions of $D = 11$ supergravity was there discussed. In particular their holonomy and Killing spinors were calculated explicitly in [50], showing that for $(p, q, r) = (1, 1, 1)$ there is $\mathfrak{so}(8)$-holonomy equal to $\mathfrak{su}(3)$ and two Killing spinors, while in all the other cases all supersymmetries are broken. In the context of the AdS/CFT correspondence, the algebraic structure of the sasakian manifolds was shown to determine the form of the dual gauge theories in [42] and in that paper the gauge dual of Q^{111} was also derived. Finally the complete Kaluza-Klein spectrum of M-theory on $AdS_4 \times Q^{111}$ and its organization in $\mathrm{Osp}(2|4) \times \mathrm{SU}(2)^3$ multiplets was derived in [51]. We review here the essential steps in the geometrical construction of Q^{111} in order to calculate the explicit form of the $\mathfrak{so}(8)$ connection

We begin by writing the Maurer Cartan equations of the Lie algebra $\mathbb{G} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$, by enumerating its generators from one to ten, the first triplet e_1, e_2, e_3 being the generators of the first $\mathfrak{su}(2)$, the second triplet e_4, e_5, e_6 the generators of the second $\mathfrak{su}(2)$ and so on. The last generator e_{10} is associated with the abelian $\mathfrak{u}(1)$ algebra. Correspondingly we have:

$$\left.\begin{aligned} 0 &= de_{1+3i} + e_{2+3i} \wedge e_{3+3i} \\ 0 &= de_{2+3i} - e_{1+3i} \wedge e_{3+3i} \\ 0 &= de_{3+3i} + e_{1+3i} \wedge e_{2+3i} \end{aligned}\right\} \quad i = 0, 1, 2$$
$$0 = de_{10} \tag{C.5}$$

Next we perform a change of basis in the above 10-dimensional algebra introducing the following new set of 1-forms:

$$
\begin{aligned}
\Sigma_1 &= \frac{1}{4\sqrt{2}\,e} e_1; \qquad \Sigma_2 = \frac{1}{4\sqrt{2}\,e} e_2 \\
\Sigma_3 &= \frac{1}{4\sqrt{2}\,e} e_4; \qquad \Sigma_4 = \frac{1}{4\sqrt{2}\,e} e_5 \\
\Sigma_5 &= \frac{1}{4\sqrt{2}\,e} e_7; \qquad \Sigma_6 = \frac{1}{4\sqrt{2}\,e} e_8 \\
\Sigma_7 &= \frac{1}{8\,e} (e_3 + e_6 + e_9 + e_{10}) \\
\Sigma_8 &= \frac{1}{2} (e_3 - e_6 - e_9 + e_{10}) \\
\Sigma_9 &= \frac{1}{2} (-e_3 + e_6 - e_9 + e_{10}) \\
\Sigma_{10} &= \frac{1}{2} (-e_3 - e_6 + e_9 + e_{10})
\end{aligned}
\tag{C.6}
$$

The meaning of the above rearrangement is the following. Apart from the rescaling by the factor $\frac{1}{4\sqrt{2}\,e}$ the first six generators are, two by two, the vielbeins of the three copies of the 2-dimensional projective space $\mathbb{P}^1 \sim \mathrm{SU}(2)/\mathrm{U}(1)$. The last four generators correspond to an orthogonal basis in the space spanned by the four Cartan generators, such that the first element in the basis is dual to the generator Z of eq. (C.4) with $p = q = r = 1$. In this way Σ_7 can be identified as the 7th-vielbein of Q^{111}. The remaining three 1-forms $\Sigma_{8,9,10}$ provide a basis for the H-subalgebra $\mathbb{H} = \mathfrak{u}(1) \oplus \mathfrak{u}(1) \oplus \mathfrak{u}(1)$. The rescalings of the vielbeins have being chosen in such a way as to produce a diagonal Ricci tensor with 7-eigenvalues all equal to $12e^2$ as it is required in order for the manifold to be a solution of $D = 11$ supergravity. Here as above e denotes the Freund Rubin parameter.

Writing the Maurer Cartan equations (C.5) in the new basis the Maurer Cartan equations (C.5) we can use them to calculate the spin connection $B^{\alpha\beta}$ of the 7-manifold by setting:

$$
\mathcal{B}^\alpha = \{\Sigma_1 \,, \ldots \Sigma_7\}
\tag{C.7}
$$

and implementing the vanishing of the torsion:

$$
d\mathcal{B}^\alpha + \mathcal{B}^{\alpha\beta} \wedge \mathcal{B}^\beta = 0
\tag{C.8}
$$

This leads to the calculation of the Riemann tensor and of the Ricci tensor:

$$
\mathcal{R}^\alpha_\beta = 12e^2\, \delta^\alpha_\beta
\tag{C.9}
$$

as required.

The connection on the $\mathfrak{so}(8)$-bundle can now be easily calculated. From its definition:

$$
\mathbf{U} = -\frac{1}{4} \mathcal{B}^{\alpha\beta}\, \tau_{\alpha\beta} - e\,\mathcal{B}^\alpha\, \tau_\alpha
\tag{C.10}
$$

we can obtain its explicit form, provided we use an explicit representation of the τ-matrices, satisfying the Clifford algebra (B.1). In appendix B.1 we displayed an explicit realization of the τ matrices which is well adapted to the discussion of the N^{010} manifold and is particularly simple. Certainly we could use such a basis also for the Q^{111}-manifold, yet, in this case it is convenient to use another basis τ'_α , related to the τ_α of eqs. (B.2) by an orthogonal SO(8) transformation:

$$\tau'_\alpha = \mathbf{O}\,\tau_\alpha\,\mathbf{O}^T \tag{C.11}$$

where:

$$\mathbf{O} = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} \\ 0 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} \end{pmatrix} \tag{C.12}$$

If the τ_α used in eq. (C.10) are the τ'_α, defined in eq. (C.12), we get a block-diagonal structure for the $\mathbf{U}$-matrix:

$$\mathbf{U} = \left(\begin{array}{c|c} \mathbf{U}_2 & 0 \\ \hline 0 & \mathbf{U}_6 \end{array}\right) \tag{C.13}$$

where:

$$\begin{aligned} \mathfrak{so}(2) \ni \mathbf{U}_2 &= \begin{pmatrix} 0 & e\,\Sigma_7 + \frac{\Sigma_8}{4} + \frac{\Sigma_9}{4} + \frac{\Sigma_{10}}{4} \\ -e\,\Sigma_7 - \frac{\Sigma_8}{4} - \frac{\Sigma_9}{4} - \frac{\Sigma_{10}}{4} & 0 \end{pmatrix} \\ &= \frac{1}{2}\begin{pmatrix} 0 & e_{10} \\ -e_{10} & 0 \end{pmatrix} \end{aligned} \tag{C.14}$$

and

$$\mathfrak{so}(6) \ni \mathbf{U}_6 = \begin{pmatrix} 0 & \frac{-e_4}{2\sqrt{2}} & \frac{e_5}{2\sqrt{2}} & \frac{e_7-e_8}{4} & \frac{-e_3+e_6+e_9}{2} & \frac{-e_7-e_8}{4} \\ \frac{e_4}{2\sqrt{2}} & 0 & \frac{e_3+e_6-e_9}{2} & \frac{e_1+e_2}{4} & \frac{-e_5}{2\sqrt{2}} & \frac{e_1-e_2}{4} \\ \frac{-e_5}{2\sqrt{2}} & \frac{-e_3-e_6+e_9}{2} & 0 & \frac{e_1-e_2}{4} & \frac{-e_4}{2\sqrt{2}} & \frac{-e_1-e_2}{4} \\ \frac{-e_7+e_8}{4} & \frac{-e_1-e_2}{4} & \frac{-e_1+e_2}{4} & 0 & \frac{e_7+e_8}{4} & \frac{e_3-e_6+e_9}{2} \\ \frac{e_3-e_6-e_9}{2} & \frac{e_5}{2\sqrt{2}} & \frac{e_4}{2\sqrt{2}} & \frac{-e_7-e_8}{4} & 0 & \frac{-e_7+e_8}{4} \\ \frac{e_7+e_8}{4} & \frac{-e_1+e_2}{4} & \frac{e_1+e_2}{4} & \frac{-e_3+e_6-e_9}{2} & \frac{e_7-e_8}{4} & 0 \end{pmatrix} \tag{C.15}$$

C.2 The N^{010} tri-sasakian manifold

The space N^{010} can be simply defined as the coset space

$$\frac{\mathcal{S}}{\mathcal{R}} = \frac{\mathrm{SU}(3)}{\mathrm{U}(1)}\,, \tag{C.16}$$

where, using the Gell-Mann matrices $\lambda^{\overline{A}}$ as $\mathfrak{su}(3)$ generators, the quotient is taken with respect to the U(1) subgroup generated by λ^8. The space N^{010}, an instance in the series of 7-dimensional coset spaces named $N^{p,q,r}$ in the classification of [34], is the only 7-dimensional coset that, when used as a compactification manifold for 11D supergravity, can preserve $\mathcal{N} = 3$ supersymmetry [54]. The complete KK spectrum of the N^{010} compactification was derived in [49], and its Osp(3|4) multiplet structure elucidated in [41, 55].

The isotropy group of N^{010} is $\mathrm{SU}(3) \times \mathrm{SU}(2)$; the SU(2) factor is the normalizer of the U(1) action and, explicitly, it is generated by $\lambda^{1,2,3}$.

In this case the underlined capital latin indices from the beginning of the alphabet run on eight values and span the adjoint representation of the $\mathfrak{su}(3)$ Lie algebra.

Let

$$\Sigma^{\overline{A}} = (\Sigma^{\alpha}, \Sigma^{8}) \tag{C.17}$$

be the Maurer-Cartan forms for $\mathfrak{su}(3)$, namely let

$$\Sigma = \frac{\mathrm{i}}{2}\, \Sigma^{\overline{A}} \lambda_{\overline{A}} = g^{-1} dg \quad ; \quad g \in \mathrm{SU}(3) \tag{C.18}$$

so that the Maurer Cartan equations

$$d\Sigma + \Sigma \wedge \Sigma = 0 \tag{C.19}$$

rewritten in the Gell-Mann basis:

$$d\Sigma^{\overline{A}} + \frac{1}{2}\, f^{\overline{A}}{}_{\overline{BC}}\, \Sigma^{\overline{B}} \wedge \Sigma^{\overline{C}} = 0 \tag{C.20}$$

define the structure constants $f^{\overline{A}}{}_{\overline{BC}}$ of the $\mathfrak{su}(3)$ Lie algebra. The vielbein corresponding to a generic $\mathrm{SU}(3) \times \mathrm{SU}(2)$-invariant metric are obtained from the coset vielbein Σ^{α} ($\alpha = 1, \ldots 7$) by rescaling independently the two groups associated to $\lambda^{\dot{\alpha}}$ ($\dot{\alpha} = 1, 2, 3$) and $\lambda^{\widetilde{\alpha}}$ ($\widetilde{\alpha} = 4, 5, 6, 7$). Indeed such a decomposition is respected both by the U(1) quotient and by the SU(2) action. Thus we have:[3]

$$\mathcal{B}^{\alpha} = (\alpha^{-1} \Sigma^{\dot{\alpha}}, \beta^{-1} \Sigma^{\widetilde{\alpha}})\ . \tag{C.21}$$

The spin connection $\mathcal{B}^{\alpha\beta}$ and the curvature associated to these vielbein are straightforwardly computed (see [56]).

The "standard" N^{010} metric is obtained with the following rescalings:

$$\alpha = -4\ e\,, \qquad \beta = \pm 4\sqrt{2}\ e\ . \tag{C.22}$$

It preserves $\mathcal{N} = 3$ supersymmetry. It is known [54] that, when N^{010} is realized as the coset (C.16), its Killing spinors must actually be *constant*. With the rescalings (C.22), there are 3 independent constant spinors η^A ($A = 1, 2, 3$) that satisfy eq. (3.31), namely

$$-\frac{1}{4} B^{\alpha\beta}{}_{\gamma} \tau_{\alpha\beta}\, \eta^A = e\, \tau_{\gamma}\, \eta^A\ . \tag{C.23}$$

They transform as a *triplet* under the SU(2) part of the isometry, which therefore truly acquires the role of the R-symmetry group $\mathrm{SU}(2)_R$ for the 4-dimensional gauged supergravity that arises from the compactification.

[3] Due to a different choice of structure constant, our rescaling α is minus twice the one used in [56].

There is a possible solution that differs from (C.22) only by the sign of the rescaling α. While the sign of β is irrelevant, because β appears quadratically also in the spin connection, reversing the sign of α amounts to reversing the sign of the spin connection (or, equivalently, to changing the orientation of the manifold). This solution with opposite orientation preserves no supersymmetry.

In the case with preserved $\mathcal{N}=3$ supersymmetry let us calculate the $\mathfrak{so}(8)$ connection as defined by eq. (4.5):

$$U^{\mathrm{SO}(8)} \equiv -\frac{1}{4}\mathcal{B}^{\alpha\beta}\tau_{\alpha\beta} - e\,\tau_\gamma\,\mathcal{B}^\gamma \tag{C.24}$$

We find its explicit expression as an 8×8 matrix:

$$\mathbf{U}^{\mathrm{SO}(8)} = \left(\begin{array}{cc|ccccc|c}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 4\,\mathcal{B}^5 & -4\,\mathcal{B}^4 & -4\,\mathcal{B}^7 & 4\,\mathcal{B}^6 & 0 \\
0 & 0 & -4\,\mathcal{B}^5 & 0 & \frac{-(\sqrt{3}\,\Sigma^8)}{2} + 2\,\mathcal{B}^3 & -2\,\mathcal{B}^2 & -2\,\mathcal{B}^1 & 0 \\
0 & 0 & 4\,\mathcal{B}^4 & \frac{\sqrt{3}\,\Sigma^8}{2} - 2\,\mathcal{B}^3 & 0 & 2\,\mathcal{B}^1 & -2\,\mathcal{B}^2 & 0 \\
0 & 0 & 4\,\mathcal{B}^7 & 2\,\mathcal{B}^2 & -2\,\mathcal{B}^1 & 0 & \frac{-(\sqrt{3}\,\Sigma^8)}{2} - 2\,\mathcal{B}^3 & 0 \\
0 & 0 & -4\,\mathcal{B}^6 & 2\,\mathcal{B}^1 & 2\,\mathcal{B}^2 & \frac{\sqrt{3}\,\Sigma^8}{2} + 2\,\mathcal{B}^3 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right) \tag{C.25}$$

where $\mathcal{B}^\alpha$ is the vielbein defined with the appropriate rescalings already included and Σ^8 is the H-connection, namely the component along λ^8 of the left-invariant 1-form Σ on the coset.

It is visually evident from eq. (C.25) that the three Killing spinors are

$$\begin{aligned}
\eta_1 &= (1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ 0) \\
\eta_2 &= (0\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ \ 0) \\
\eta_3 &= (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 1)
\end{aligned} \tag{C.26}$$

Since the non trivial part of the operator $\mathbf{U}^{\mathrm{SO}(8)}$ is only the block in the five directions $2,3,4,5,6$.

Hence, in this case, we have a natural way of introducing an orthogonal basis of sections of the $\mathfrak{so}(8)$ spinor bundle. We use $\eta_{\underline{A}}$ as three basis vectors, while the other five can be chosen to be

$$\xi_i = \overrightarrow{\epsilon}_{i+1} \quad ; \quad i = 1,\dots,5 \tag{C.27}$$

where $\overrightarrow{\epsilon}_i$ are the standard orthonormal euclidean vectors in eight dimensions.

With this choice the 1-form connection $\mathbf{U}_{AB}$ is just 8×8 matrix $\mathbf{U}^{\mathfrak{so}(8)}$ as given in eq. (C.25).

References

[1] N. Berkovits, *Covariant quantization of the supermembrane*, *JHEP* **09** (2002) 051 [hep-th/0201151].

[2] E. Cremmer and B. Julia, *Supergravity theory in eleven dimensions*, *Phys. Lett.* **B 76** (1978) 409; *The* SO(8) *supergravity*, *Nucl. Phys.* **B 159** (1979) 141.

[3] R. D'Auria and P. Fré, *Geometric supergravity in* $D = 11$ *and its hidden supergroup*, *Nucl. Phys.* **B 201** (1982) 101.

[4] P. Fré' and P.A. Grassi, *Pure spinors, free differential algebras and the supermembrane*, *Nucl. Phys.* **B 763** (2007) 1 [hep-th/0606171].

[5] N. Berkovits, *Super-Poincaré covariant quantization of the superstring*, *JHEP* **04** (2000) 018 [hep-th/0001035].

[6] N. Berkovits and P.S. Howe, *Ten-dimensional supergravity constraints from the pure spinor formalism for the superstring*, *Nucl. Phys.* **B 635** (2002) 75 [hep-th/0112160].

[7] H. Ooguri, J. Rahmfeld, H. Robins and J. Tannenhauser, *Holography in superspace*, *JHEP* **07** (2000) 045 [hep-th/0007104].

[8] P.A. Grassi and L. Tamassia, *Vertex operators for closed superstrings*, *JHEP* **07** (2004) 071 [hep-th/0405072].

[9] D. Tsimpis, *Curved* $11D$ *supergeometry*, *JHEP* **11** (2004) 087 [hep-th/0407244].

[10] K. Peeters, P. Vanhove and A. Westerberg, *Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace*, *Class. and Quant. Grav.* **18** (2001) 843 [hep-th/0010167].

[11] G. Dall'Agata et al., *The* Osp(8|4) *singleton action from the supermembrane*, *Nucl. Phys.* **B 542** (1999) 157 [hep-th/9807115].

[12] I. Pesando, *A kappa gauge fixed type IIB superstring action on* $AdS_5 \times S^5$, *JHEP* **11** (1998) 002 [hep-th/9808020].

[13] R. Kallosh and A.A. Tseytlin, *Simplifying superstring action on* $AdS_5 \times S^5$, *JHEP* **10** (1998) 016 [hep-th/9808088].

[14] N.J. Berkovits and J.M. Maldacena, $N = 2$ *superconformal description of superstring in Ramond-Ramond plane wave backgrounds*, *JHEP* **10** (2002) 059 [hep-th/0208092].

[15] N. Berkovits, $N = 2$ σ*-models for Ramond-Ramond backgrounds*, *JHEP* **10** (2002) 071 [hep-th/0210078].

[16] N. Berkovits, *Pure spinor formalism as an* $N = 2$ *topological string*, *JHEP* **10** (2005) 089 [hep-th/0509120].

[17] E. Witten, *Perturbative gauge theory as a string theory in twistor space*, *Commun. Math. Phys.* **252** (2004) 189 [hep-th/0312171].

[18] P.A. Grassi and P. van Nieuwenhuizen, *Gauging cosets*, *Nucl. Phys.* **B 702** (2004) 189 [hep-th/0403209].

[19] P.A. Grassi, G. Policastro, M. Porrati and P. Van Nieuwenhuizen, *Covariant quantization of superstrings without pure spinor constraints*, *JHEP* **10** (2002) 054 [hep-th/0112162].

[20] P.A. Grassi, G. Policastro and P. van Nieuwenhuizen, *The massless spectrum of covariant superstrings*, *JHEP* **11** (2002) 001 [hep-th/0202123].

[21] R. D'Auria and P. Fré, *Geometric supergravity in* $D = 11$ *and its hidden supergroup*, *Nucl. Phys.* **B 201** (1982) 101.

[22] P. Fré, *Comments on the six index photon in $D = 11$ supergravity and the gauging of free differential algebras*, *Class. and Quant. Grav.* **1** (1984) L81.

[23] L. Castellani, R. D'Auria and P. Fré, *Supergravity and superstrings: a geometric perspective*, World Scientific, Singapore (1991).

[24] E. Cremmer and S. Ferrara, *Formulation of eleven-dimensional supergravity in superspace*, *Phys. Lett.* **B 91** (1980) 61.

[25] P. Kaste, R. Minasian and A. Tomasiello, *Supersymmetric M-theory compactifications with fluxes on seven-manifolds and G-structures*, *JHEP* **07** (2003) 004 [hep-th/0303127].

[26] L. Castellani, R. D'Auria and P. Fré, $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ *from $D = 11$ supergravity*, *Nucl. Phys.* **B 239** (1984) 610.

[27] P.G.O. Freund and M.A. Rubin, *Dynamics of dimensional reduction*, *Phys. Lett.* **B 97** (1980) 233.

[28] A. Bilal, J.-P. Derendinger and K. Sfetsos, *(weak) G_2 holonomy from self-duality, flux and supersymmetry*, *Nucl. Phys.* **B 628** (2002) 112 [hep-th/0111274].

[29] R. D'Auria and P. Fré, *On the fermion mass spectrum of kaluza klein supergravity*, *Ann. Phys. (NY)* **157** (1984) 1.

[30] F. Englert, *Spontaneous compactification of eleven-dimensional supergravity*, *Phys. Lett.* **B 119** (1982) 339.

[31] M.A. Awada, M.J. Duff and C.N. Pope, *$N = 8$ supergravity breaks down to $N = 1$*, *Phys. Rev. Lett.* **50** (1983) 294.

[32] R. D'Auria, P. Fré and P. van Nieuwenhuizen, *$N = 2$ matter coupled supergravity from compactification on a coset G/H possessing an additional Killing vector*, *Phys. Lett.* **B 136** (1984) 347.

[33] L. Castellani and L.J. Romans, *$N = 3$ and $N = 1$ supersymmetry in a new class of solutions for $D = 11$ supergravity*, *Nucl. Phys.* **B 238** (1984) 683.

[34] L. Castellani, L.J. Romans and N.P. Warner, *A classification of compactifying solutions for $D = 11$ supergravity*, *Nucl. Phys.* **B 241** (1984) 429.

[35] D.Z. Freedman and H. Nicolai, *Multiplet shortening in* $\mathrm{Osp}(N|4)$, *Nucl. Phys.* **B 237** (1984) 342.

[36] A. Ceresole, P. Fré and H. Nicolai, *Multiplet structure and spectra of $N = 2$ supersymmetric compactifications*, *Class. and Quant. Grav.* **2** (1985) 133.

[37] A. Casher, F. Englert, H. Nicolai and M. Rooman, *The mass spectrum of supergravity on the round seven sphere*, *Nucl. Phys.* **B 243** (1984) 173.

[38] M.J. Duff, B.E.W. Nilsson and C.N. Pope, *Kaluza-Klein supergravity*, *Phys. Rept.* **130** (1986) 1.

[39] M. Billó, D. Fabbri, P. Fré, P. Merlatti and A. Zaffaroni, *Shadow multiplets in AdS_4/CFT_3 and the super-Higgs mechanism*, *Nucl. Phys.* **B 591** (2000) 139 [hep-th/0005220].

[40] M. Billó, D. Fabbri, P. Fré, P. Merlatti and A. Zaffaroni, *Rings of short $N = 3$ superfields in three dimensions and M-theory on $AdS_4 \times N^{(0,1,0)}$*, *Class. and Quant. Grav.* **18** (2001) 1269 [hep-th/0005219].

[41] P. Fré', L. Gualtieri and P. Termonia, *The structure of* $N = 3$ *multiplets in* AdS_4 *and the complete* $\mathrm{Osp}(3|4) \times \mathrm{SU}(3)$ *spectrum of M-theory on* $AdS_4 \times N^{(0,1,0)}$, *Phys. Lett.* **B 471** (1999) 27 [hep-th/9909188].

[42] D. Fabbri et al., *3d superconformal theories from sasakian seven-manifolds: new nontrivial evidences for* AdS_4/CFT_3, *Nucl. Phys.* **B 577** (2000) 547 [hep-th/9907219].

[43] D. Fabbri, P. Fré, L. Gualtieri and P. Termonia, *M-theory on* $AdS_4 \times M^{(111)}$*: the complete* $\mathrm{Osp}(2|4) \times \mathrm{SU}(3) \times \mathrm{SU}(2)$ *spectrum from harmonic analysis*, *Nucl. Phys.* **B 560** (1999) 617 [hep-th/9903036].

[44] R. D'Auria and P. Fré *Universal Bose-Fermi mass-relations in Kaluza-Klein supergravity and harmonic analysis on coset manifolds with Killing spinors*, *Ann. Phys. (NY)* **162** (1985) 372.

[45] W. Heidenreich, *All linear unitary irreducible representations of de Sitter supersymmetry with positive energy*, *Phys. Lett.* **B 110** (1982) 461.

[46] R. D'Auria and P. Fré, *Spontaneous generation of* $\mathrm{Osp}(4|8)$ *symmetry in the spontaneous compactification of* $D = 11$ *supergravity*, *Phys. Lett.* **B 121** (1983) 141.

[47] P. Fré, *Gaugings and other supergravity tools of p-brane physics*, hep-th/0102114.

[48] L. Castellani et al., *G/H M-branes and* $AdS(p + 2)$ *geometries*, *Nucl. Phys.* **B 527** (1998) 142 [hep-th/9803039].

[49] P. Termonia, *The complete* $N = 3$ *Kaluza-Klein spectrum of* $11D$ *supergravity on* $AdS_4 \times N^{(0,1,0)}$, *Nucl. Phys.* **B 577** (2000) 341 [hep-th/9909137].

[50] R. D'Auria, P. Fré and P. van Nieuwenhuizen, $N = 2$ *matter coupled supergravity from compactification on a coset G/H possessing an additional Killing vector*, *Phys. Lett.* **B 136** (1984) 347.

[51] P. Merlatti, *M-theory on* $AdS_4 \times Q^{(1,1,1)}$*: the complete* $\mathrm{Osp}(2|4) \times \mathrm{SU}(2) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$ *spectrum from harmonic analysis*, *Class. and Quant. Grav.* **18** (2001) 2797 [hep-th/0012159].

[52] P. Pasti, D.P. Sorokin and M. Tonin, *On gauge-fixed superbrane actions in AdS superbackgrounds*, *Phys. Lett.* **B 447** (1999) 251 [hep-th/9809213].

[53] P. Pasti, D.P. Sorokin and M. Tonin, *Branes in super-AdS backgrounds and superconformal theories*, hep-th/9912076.

[54] L. Castellani and L.J. Romans, $N = 3$ *and* $N = 1$ *supersymmetry in a new class of solutions for* $D = 11$ *supergravity*, *Nucl. Phys.* **B 238** (1984) 683;
L. Castellani, *Fermions with nonzero* $\mathrm{SU}(3)$ *triality in the* $M(pqr)$ *and* $N(pqr)$ *solutions of* $D = 11$ *supergravity*, *Class. and Quant. Grav.* **1** (1984) L97.

[55] P. Fré', L. Gualtieri and P. Termonia, *The structure of* $N = 3$ *multiplets in* AdS_4 *and the complete* $\mathrm{Osp}(3|4) \times \mathrm{SU}(3)$ *spectrum of M-theory on* $AdS_4 \times N^{(0,1,0)}$, *Phys. Lett.* **B 471** (1999) 27 [hep-th/9909188].

[56] L. Castellani, *On G/H geometry and its use in M-theory compactifications*, *Ann. Phys. (NY)* **287** (2001) 1 [hep-th/9912277].